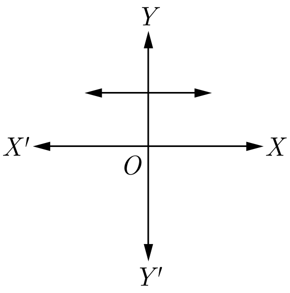
CHAPTER 2

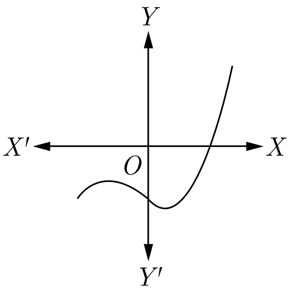
POLYNOMIALS

EXERCISE 1.1

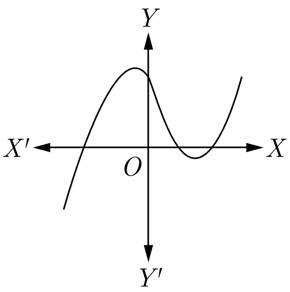
1. The graph of $y = p(x)$ are given in Fig. below, for some polynomials $p(x)$. Find the number of zeroes of $p(x)$, in each case.



(i)



(ii)



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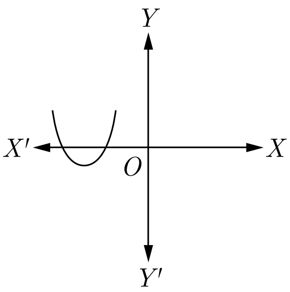

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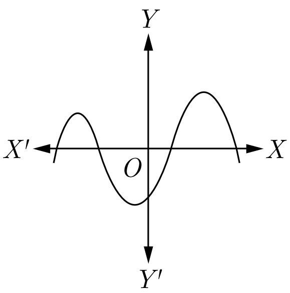

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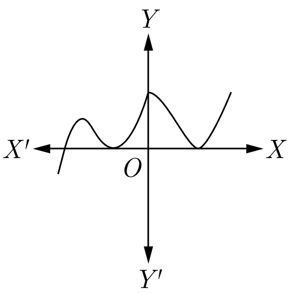


(vi)

Sol :

(i) Here, there are no zeroes as the graph does not intersect the x-axis.

(ii) Here, the number of zeroes is 1 as the graph intersects the x-axis at one point only.

(iii) Here, the number of zeroes is 3 as the graph intersects the x-axis at three points only.

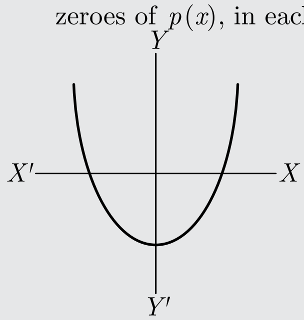
(iv) Here, the number of zeroes is 2 as the graph intersects the x-axis at two points only.

(v) Here, the number of zeroes is 4 as the graph intersects the x-axis at four points only.

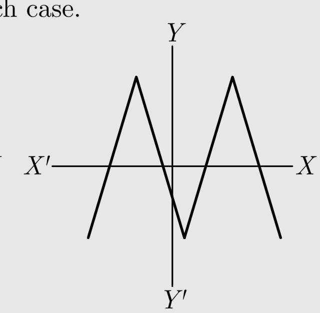
(vi) Here, the number of zeroes is 3 as the graph intersects the x-axis at three points only.

PRACTICE :

1. The graph of $y = p(x)$ are given in Fig. below, for some polynomials $p(x)$ Find the number of zeroes of $p(x)$, in each case.



(i)



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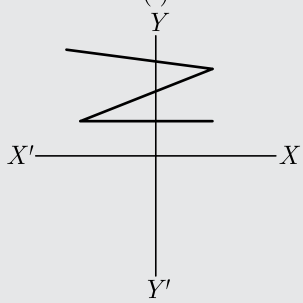

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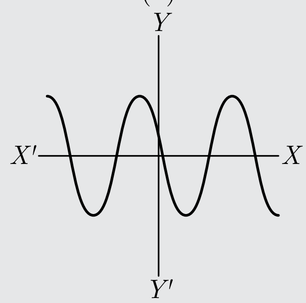

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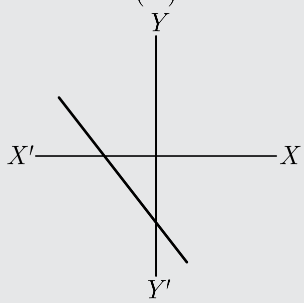
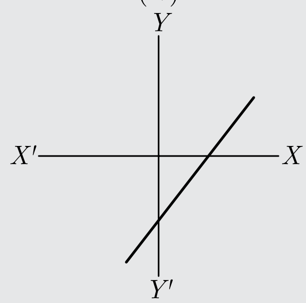


(v)

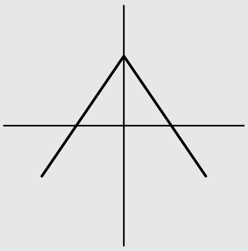


(vi)

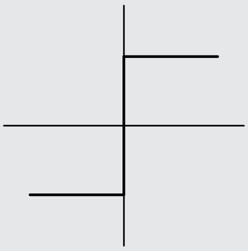
Ans : (i) 2, (ii) 4, (iii) 0, (iv) 6, (v) 1, (vi) 1

PRACTICE :

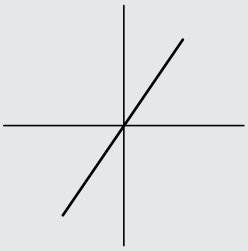
2. The graph of $y = p(x)$ are given in Fig. below, for some polynomials $p(x)$ Find the number of zeroes of $p(x)$, in each case.



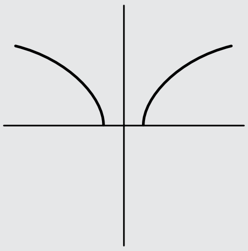
(i)



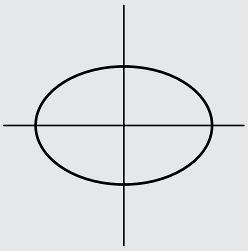
(ii)



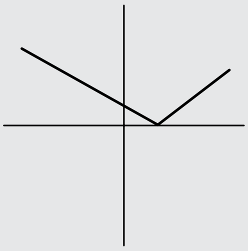
(iii)



(iv)



(v)



(vi)

Ans : (i) 2, (ii) 0, (iii) 1, (iv) 2, (v) 2, (vi) 1

EXERCISE 1.2

1. Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.

(i) $x^2 - 2x - 8$ (ii) $4s^2 - 4s + 1$

(iii) $6x^2 - 3 - 7x$ (iv) $4u^2 + 8u$

(v) $t^2 - 15$ (vi) $3x^2 - x - 4$

Sol :

(i)
$$\begin{aligned} x^2 - 2x - 8 &= x^2 + 2x - 4x - 8 \\ &= x(x+2) - 4(x+2) \\ &= (x-2)(x-4) \end{aligned}$$

The value of $x^2 - 2x - 8$ will be zero if the value of $(x+2)(x-4)$ is zero, i.e.,

$$x + 2 = 0 \Rightarrow x = -2$$

or
$$x - 4 = 0 \Rightarrow x = 4$$

Therefore the zeroes of $x^2 - 2x - 8$ are -2 and 4.

Now, Sum of the zeroes

$$= (-2) + 4 = 2 = \frac{-(-2)}{1} = \frac{-\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

and product of the zeroes

$$= (-2)(4) = -8 = \frac{-8}{1} = \frac{\text{Cons tant term}}{\text{Coefficient of } x^2}$$

(ii)
$$\begin{aligned} 4s^2 - 4s + 1 &= 4s^2 - 2s - 2s + 1 \\ &= 2s(2s-1) - 1(2s-1) \\ &= (2s-1)(2s-1) \end{aligned}$$

Therefore, the value of $4s^2 - 4s + 1$ will be zero if the value of $(2s-1)(2s-1)$ is zero.

i.e.,
$$2s - 1 = 0 \Rightarrow s = \frac{1}{2}$$

Hence the zeroes of $4s^2 - 4s + 1$ are $\frac{1}{2}$ and $\frac{1}{2}$.

Now, sum of the zeroes

$$= \frac{1}{2} + \frac{1}{2} = 1 = \frac{-(-4)}{4} = \frac{-\text{Coefficient of } s}{\text{Coefficient of } s^2}$$

and product of the zeroes

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} = \frac{\text{Constant term}}{\text{Coefficient of } s^2}$$

(iii)
$$\begin{aligned} 6x^2 - 3 - 7x &= 6x^2 - 7x - 3 \\ &= 6x^2 - 9x + 2x - 3 \\ &= 3x(2x-3) + 1(2x-3) \\ &= (3x+1)(2x-3) \end{aligned}$$

So, the value of $6x^2 - 3 - 7x$ will be zero if the value of $(3x+1)(2x-3)$ is zero, i.e.,

$$3x + 1 = 0 \Rightarrow x = -\frac{1}{3}$$

or
$$2x - 3 = 0 \Rightarrow x = \frac{3}{2}$$

Hence, the zeroes of $6x^2 - 3 - 7x$ are $-\frac{1}{3}$ and $\frac{3}{2}$

Now sum of the zeroes

$$= -\frac{1}{3} + \frac{3}{2} = \frac{7}{6} = \frac{-(-7)}{6} = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

and product of the zeroes

$$= \left(-\frac{1}{3}\right)\left(\frac{3}{2}\right) = \frac{-3}{6} = \frac{\text{Cons tant term}}{\text{Coefficient of } x^2}$$

(iv)
$$4u^2 + 8u = 4u(u+2)$$

The value of $4u^2 + 8u$ will be zero if the value of $4u(u+2)$ is zero, i.e.,

$$u = 0$$

or
$$u + 2 = 0 \Rightarrow u = -2$$

Hence, the zeroes of $4u^2 + 8u$ are 0 and -2.

Now, sum of the Zeroes

$$= 0 + (-2) = -2 = \frac{-8}{4} = \frac{-\text{Coefficient of } u}{\text{Coefficient of } u^2}$$

and product of the zeroes

$$= (0)(-2) = 0 = \frac{\text{Constant term}}{\text{Coefficient of } u^2}$$

(v) $t^2 - 15 = (t - \sqrt{15})(t + \sqrt{15})$

The value of $t^2 - 15$ will be zero if the value of $(t - \sqrt{15})(t + \sqrt{15})$ is zero.

i.e., $t - \sqrt{15} = 0 \Rightarrow t = \sqrt{15}$

or $t + \sqrt{15} = 0 \Rightarrow t = -\sqrt{15}$

Hence, the zeroes of $t^2 - 15$ are $\sqrt{15}$ and $-\sqrt{15}$

Now, sum of the zeroes

$$= \sqrt{15} + (-\sqrt{15}) = 0 = \frac{0}{1} = \frac{-\text{Coefficient of } t}{\text{Coefficient of } t^2}$$

and product of the zeroes

$$= (\sqrt{15})(-\sqrt{15}) = -15 = \frac{-15}{1} = \frac{\text{Constant term}}{\text{Coefficient of } t^2}$$

(vi) $3x^2 - x - 4 = 3x^2 + 3x - 4x - 4$

$$= 3x(x+1) - 4(x+1)$$

$$= (x+1)(3x-4)$$

The value of $3x^2 - x - 4$ will be zero if the value of $(x+1)(3x-4)$ is zero.

$$x + 1 = 0 \Rightarrow x = -1$$

or $3x - 4 = 0 \Rightarrow x = \frac{4}{3}$

Hence, the zeroes of $3x^2 - x - 4$ are -1 and $\frac{4}{3}$.

Now, sum of the zeroes

$$= -1 + \frac{4}{3} = \frac{1}{3} \quad \frac{-(-1)}{3} = \frac{-\text{Coeffecient of } x}{\text{Coeffiecient of } x^2}$$

and, product of the zeroes

$$= (-1)\left(\frac{4}{3}\right) = -\frac{4}{3} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

PRACTICE :

1. Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.

(i) $x^2 - 1$ (ii) $x^2 - 3x - 2$

(iii) $2x^2 + 4x$ (iv) $t^2 - 4t + 3$

(v) $s^2 + s - 2$ (vi) $t^2 - 16$

(vii) $x^2 + 2\sqrt{2}\,x - 6$

(viii) $\sqrt{3}\,x^2 + 10x + 7\sqrt{3}$

(ix) $x^2 - (\sqrt{3} + 1)x + \sqrt{3}$

(x) $a(x^2 + 1) - x(a^2 + 1)$

Ans : (i) $1, -1$ (ii) $1, 2$ (iii) $0, -2$ (iv) $1, 3$ (v) $-2, 1$ (vi) $4, -4$ (vii) $\sqrt{2}, -3\sqrt{2}$ (viii) $-\sqrt{3}, \frac{-7}{\sqrt{3}}$ (ix) $\sqrt{3}, 1$ (x) $a, \frac{1}{a}$

2. Find a quadratic polynomial each with the given numbers as the sum and product of its zeroes respectively.

(i) $\frac{1}{4}, -1$ (ii) $\sqrt{2}, \frac{1}{3}$

(iii) $0, \sqrt{5}$ (iv) $1, 1$

(v) $-\frac{1}{4}, \frac{1}{4}$ (vi) $4, 1$

Sol :

(i) $\frac{1}{4}, -1$

Let the polynomial be $ax^2 + bx + c$ and its zeroes be α and β.

Then, $\alpha + \beta = \frac{1}{4} = \frac{-b}{a}$

and $\alpha\beta = -1 = \frac{-4}{4} = \frac{c}{\alpha}$

If $\alpha = 4$, $b = -1$ and $c = -4$, the quadratic polynomial will be

$4x^2 - x - 4$.

Alternative :

Sum of zeroes $= \frac{1}{4}$

Product of zeroes $= -1$

Polynomial $= k(x^2 - \text{sum } x + \text{product})$

$$k\left(x^2 - \frac{1}{4}x + (-1)\right)$$

$$k\left[x^2 - \frac{1}{4}x - 1\right] = k\left[\frac{4x^2 - x - 1}{4}\right]$$

$$\frac{k}{4} = (4x^2 - x - 1)$$

$$= 4x^2 - x - 1 \text{ (here } k = 4)$$

(ii) $\sqrt{2}, \frac{1}{3}$

Let the polynomial be $zx^2 + bx + c$ and its zeroes be α and β.

Then, $\alpha + \beta = \sqrt{2} = \frac{3\sqrt{2}}{3} = \frac{-b}{a}$

and $\alpha\beta = \frac{1}{3} = \frac{c}{a}$

If $a = 3$, $b = -3\sqrt{2}$ and $c = 1$, then quadratic polynomial will be $3x^2 - 3\sqrt{2}\,x + 1$.

(iii) $0, \sqrt{5}$

Let the polynomial be $ax^2 + bx + c$ and its zeroes be α and β.

Then, $\alpha + \beta = 0 = \frac{0}{1} = \frac{-b}{a}$

and $\alpha\beta = \sqrt{5} = \frac{\sqrt{5}}{1} = \frac{c}{a}$

If $a = 1$, $b = 0$ and $c = \sqrt{5}$, the quadratic polynomial will be $x^2 - 0x + \sqrt{5}$ or $x^2 + \sqrt{5}$.

Alternative :

Sum of zeroes $= 0$

Product of zeroes $= \sqrt{5}$

Polynomial $= k(x^2 - \text{sum}\, x + \text{product})$

$= k[x^2 - 0x + \sqrt{5}] = k(x^2 + \sqrt{5})$

Here let $k = 1$

then polynomial will be $x^2 + \sqrt{5}$

(iv) $1, 1$

Let the polynomial be $ax^2 + bx + c$ and its zeroes be α and β.

Then, $\alpha + \beta = 1 = \frac{-(-1)}{1} = \frac{-b}{a}$

and $\alpha\beta = 1 = \frac{1}{1} = \frac{c}{a}$

If $a = 1$ $b = -1$ and $c = 1$ then quadratic polynomial will be $x^2 - 4x + 1$.

(v) $-\frac{1}{4}, \frac{1}{4}$

Let the polynomial be $ax^2 + bx + c$ and its zeroes be α and β.

Then, $\alpha + \beta = -\frac{1}{4} = \frac{-b}{a}$

and $\alpha\beta = \frac{1}{4} = \frac{c}{a}$

If $a = 4$, $b = 1$ and $c = 1$, then the quadratic polynomial will be $4x^2 + x + 1$

(vi) $4, 1$

Let the polynomial be $ax^2 + bx + c$ and its zeroes be α and β.

Then, $\alpha + \beta = 4 = \frac{-b}{a}$

and $\alpha\beta = 1 = \frac{1}{1} = \frac{c}{a}$

If $a = 1$, $b = -4$ and $c = 1$, then the quadratic polynomial will be $x^2 - 4x + 1$.

Alternative :

Sum of zeroes $= 4$, product of zeroes $= 1$

polynomial $= k[x^2 - \text{sum}\, x + \text{product}]$

$= k[x^2 - 4x + 1]$

$= x^2 - 4x + 1$ [here let $k = 1$]

PRACTICE :

1. Find a quadratic polynomial each with the given numbers as the sum and product of its zeroes respectively.

(i) $2, -3$ (ii) $\sqrt{3}, -\sqrt{3}$

(iii) $0, 2$ (iv) $-3, -2$

(v) $4, 4$ (vi) $\frac{1}{2}, \frac{1}{3}$

(vii) $\sqrt{3}, \frac{1}{2}$ (vii) $5, 1$

(ix) $2, 2$ (x) $0, \sqrt{3}$

(xi) $\frac{1}{2}, -3$ (xii) $-\frac{1}{3}, \frac{1}{3}$

Ans : (i) $x^2 + x - 6$ (ii) $x^2 - 3$ (iii) $x^2 - 2x$ (iv) $x^2 + 5x + 6$ (v) $x^2 - 8x + 16$ (vi) $6x^2 - 5x + 1$ (vii) $2x^2 - (2\sqrt{3} + 1)x + \sqrt{2}$ (viii) $x^2 - 6x + 5$ (ix) $x^2 - 4x + 4$ (x) $x^2 - \sqrt{3}\,x$ (xi) $2x^2 + 5x - 3$ (xii) $9x^2 - 1$

EXERCISE 1.3

1. Divide the polynomial $p(x)$ by the polynomial $g(x)$ and find the quotient and remainder in each of the following :

(i) $p(x) = x^3 - 3x^2 5x - 3,\ g(x) = x^2 - 2$

(ii) $p(x) = x^4 - 3x^2 + 4x + 5,\ g(x) = x^2 + 1 - x$

(iii) $p(x) = x^4 - 5x + 6,\ g(x) = 2 - x^2$

Sol :

(i) $p(x) = x^3 - 3x^2 5x - 3,\ g(x) = x^2 - 2$

We have $p(x) = x^3 - 3x^2 + 5x - 3$

and $g(x) = x^2 - 2$

Let the quotient be $q(x)$ and the remainder be $r(x)$. Then by Euclid's Division Algorithm,

$$p(x) = g(x) \cdot q(x) + r(x)$$

$$q(x) = \frac{p(x)}{g(x)} - \frac{r(x)}{g(x)}$$

$$= \frac{x^3 - 3x^2 + 5x - 3}{x^2 - 2} - \frac{r(x)}{x^2 - 2}$$

$$\begin{array}{r|l} & x - 3 \\ x^2 - 2 & x^3 - 3x^2 + 5x - 3 \\ & x^3 \quad\quad - 2x \\ \hline & -3x^2 + 7x - 3 \\ & -3x^2 \quad\quad + 6 \\ \hline & 7x - 9 = \text{Remainder} \end{array}$$

Quotient $q(x) = x - 3$

Remainder $r(x) = 7x - 9$

(ii) $p(x) = x^4 - 3x^2 + 4x + 5,\ g(x) = x^2 + 1 - x$

We have $p(x) = x^4 - 3x^2 + 4x + 5$

$g(x) = x^2 + 1 - x\ \ x^2 - x + 1$

Let the quotient be $q(x)$ and the remainder be $r(x)$.

Then by Euclid's Division Algorithm,

$$p(x) = g(x)\cdot q(x) + r(x)$$

$$q(x) = \frac{p(x)}{g(x)} - \frac{r(x)}{g(x)}$$

$$= \frac{x^4-3x^2+4x+5}{x^2-x+1} - \frac{r(x)}{x^2-x+1}$$

$$= x^2+x-3-\frac{r(x)}{x^2-x+1}$$

$$\begin{array}{r|l} & x^2+x-3 \\ \hline x^2-x+1 & x^4-3x^2+4x+5 \\ & x^4-x^3+x^2 \\ \hline & -x^3-4x^2+4x \\ & -x^3+x^2+x \\ \hline & -3x^2+3x+5 \\ & -3x^2+3x-3 \\ \hline & 8 = \text{Remainder} \end{array}$$

Quotient $q(x) = x^2+x-3$

and Remainder $r(x) = 8$

(iii) $p(x) = x^4-5x+6,\ g(x) = 2-x^2$

We have $p(x) = x^4-5x+6$

$g(x) = 2-x^2$

Let the quotient be $q(x)$ and the remainder be $r(x)$. Then by Euclid's Division Algorithm,

$$p(x) = g(x)\cdot q(x) + r(x)$$

$$q(x) = \frac{p(x)}{g(x)} - \frac{r(x)}{g(x)}$$

$$= \frac{x^4-5x+6}{2-x^2} - \frac{r(x)}{2-x^2}$$

$$= \frac{x^4-5x+6}{-x^2+2} - \frac{r(x)}{-x^2+2}$$

$$= -x^2-2-\frac{r(x)}{-x^2+2}$$

$$\begin{array}{r|l} & -x^2-2 \\ \hline 2-x^2 & x^4-5x+6 \\ & x^4-2x^2 \\ \hline & 2x^2-5x+6 \\ & 2x^2-4 \\ \hline & -5x+10 = \text{Remainder} \end{array}$$

Hence, quotient $q(x) = -x^2-2$

and remainder $r(x) = -5x+10$

PRACTICE :

1. Divide the polynomial $p(x)$ by the polynomial $g(x)$ and find the quotient and remainder in each of the following :

(i) $p(x) = x^3-6x^2+11x-6,\ g(x) = x^2+x+1$

(ii) $p(x) = 10x^4+17x^3-62x^2+30x-3,$ $g(x) = 2x^2+7x+1$

(iii) $p(x) = 4x^3-8x+8x^2+7, g(x) = 2x^2-x+1$

Ans : (i) $x-7,\ 17x+1$ (ii) $5x^2-9x-2,$ $53x-1$ (iii) $2x+5,\ 11x+2$

2. Divide the polynomial $p(x)$ by the polynomial $g(x)$ and find the quotient and remainder in each of the following :

(i) $p(x) = 15x^3-20x^2+13x-12,$ $g(x) = 2-2x+x^2$

(ii) $p(x) = 14x^3-5x^2+9x-1,$ $g(x) = 2x-1$

(iii) $p(x) = 6x^3+11x^2-39x-65,$ $g(x) = x^2-1+x$

Ans : (i) $15x+10,\ 3x-32$ (ii) $7x^2+x+5,\ 4$ (iii) $6x+5,\ -38x-60$

2. Check whether the first polynomial is a factor of the second polynomial by dividing the second polynomial by the first polynomial :

(i) $t^2-3,\ 2t^4+3t^3-2t^2-9t-12$

(ii) $x^2+3x+1,\ 3x^4+5x^3-7x^2+2x+2$

(iii) $x^3-3x+1,\ x^5-4x^3+x^2+3x+1$

Sol :

(i) $t^2-3,\ 2t^4+3t^3-2t^2-9t-12$

Dividing $2t^4+3t^3-2t^2-9t-12$ by t^2-3 we have

$$\begin{array}{r|l} & 2t^2+3t+4 \\ \hline t^2-3 & 2t^4+3t^3-2t^2-9t-12 \\ & 2t^4 \quad -6t^2 \\ \hline & 3t^3+4t^2-9t-12 \\ & 3t^3 \quad -9t \\ \hline & 4t^2 \quad -12 \\ & 4t^2 \quad -12 \\ \hline & 0 \end{array}$$

Since, the remainder is zero. Therefore r^2-3 is a factor of the polynomial $2t^4+3t^3-2t^2-9t-12$

(ii) $x^2+3x+1,\ 3x^4+5x^3-7x^2+2x+2$

Dividing $3x^4+5x^3-7x^2+2x+2$ by x^2+3x+1

$$\begin{array}{r|l} & 3x^2-4x+2 \\ \hline x^2+3x+1 & 3x^4+5x^3-7x^2+2x+2 \\ & 3x^4+9x^3+3x^2 \\ \hline & -4x^3-10x^2+2x \\ & -4x^3-12x^2-4x \\ \hline & 2x^2+6x+2 \\ & 2x^2+6x+2 \\ \hline & 0 \end{array}$$

Since the remainder is zero. Therefore x^2+3x+1 is a factor of the polynomial $3x^4+5x^3-7x^2+2x+2$.

(iii) $x^3-3x+1,\ x^5-4x^3+x^2+3x+1$

Dividing $x^5-4x^3+x^2+3x+1$ by x^3-3x+1

$$\begin{array}{r|l} & x^2-1 \\ x^3-3x+1 & x^5-4x^3+x^2+3x+1 \\ & \underline{x^5-3x^3+x^2} \\ & -x^3 \qquad +3x+1 \\ & \underline{-x^3 \qquad +3x-1} \\ & \qquad\qquad 2 \end{array}$$

Here, the remainder is $2(\neq 0)$. Therefore x^3-3x+1 is not a factor of the polynomial $x^5-4x^3+x^2+3x+1$.

PRACTICE :

1. Check whether the first polynomial is a factor of the second polynomial by dividing the second polynomial by the first polynomial :
   (i) $x^2+3x+1,\ 3x^4+5x^3-7x+2x+2$
   (ii) $2x^2-x+3,\ 6x^5-x^4+4x^3-5x^2-x-15$
   (iii) $x-1,\ x^3-6x^2+11x+6$
   Ans : (i) Yes, (ii) Yes, (iii) No.
2. Check whether the first polynomial is a factor of the second polynomial by dividing the second polynomial by the first polynomial :
   (i) $x^2+2x-3,\ x^4+2x^3-2x^2+x-1$
   (ii) $x^2-4x+3,\ x^4+2x^3-13x^2-12x+21$
   (iii) $x^2-3,\ x^4-3x^3-x^2+9x-6$
   Ans : (i) No, (ii) No, (iii) Yes.

3. Obtain all other zeroes of $3x^4+6x^3-2x^2-10x-5$, if two of its zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$.

Sol :

Since two zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$, therefore

$\left(x-\sqrt{\frac{5}{3}}\right)$ and $\left(x+\sqrt{\frac{5}{3}}\right)$ will be the factors of the given polynomial.

Now, $\left(x-\sqrt{\frac{5}{3}}\right)\left(x+\sqrt{\frac{5}{3}}\right)=x^2-\frac{5}{3}=\frac{3x^2-5}{3}$

$$=\frac{1}{3}(3x^2-5)$$

$(3x^2-5)$ will be a factor of the given polynomial.
Dividing the polynomial $3x^4+6x^3-2x^2-10x-5$ by $3x^2-5$

$$\begin{array}{r|l} & x^2+2x+1 \\ 3x^2-5 & 3x^4+6x^3-2x^2-10x-5 \\ & \underline{3x^4 \qquad -5x^2} \\ & \quad 6x^3+3x^2-10x-5 \\ & \quad \underline{6x^3 \qquad -10x} \\ & \qquad 3x^2 \qquad -5 \\ & \qquad \underline{3x^2 \qquad -5} \\ & \qquad\qquad 0 \end{array}$$

Hence, $3x^4+6x^3-2x^2-10x-5$

$$=(3x^2-5)(x^2+2x+1)$$

Now, $x^2+2x+1=x^2+x+x+1$

$$=x(x+1)+1(x+1)$$

$$=(x+1)(x+1)$$

Therefore its other zeroes will be -1 and -1. Thus, all the zeroes of the bi-quadratic polynomial will be $\sqrt{\frac{5}{3}},-\sqrt{\frac{5}{3}},-1$ and -1.

PRACTICE :

1. Obtain all other zeroes of $2x^4-2x^3-7x^2+3x+6$, if two its zeroes are $-\sqrt{\frac{3}{2}}$ and $\sqrt{\frac{3}{2}}$
   Ans : 2, -1
2. Obtain all other zeroes of $x^4-3x^3-x^2+9x-6$, if two of its zeroes are $-\sqrt{3}$ and $\sqrt{3}$
   Ans : 1, 2

4. On dividing x^3-3x^2+x+2 by a polynomial $g(x)$, the quotient and remainder were $x-2$ and $-2x+4$, respectively. Find $g(x)$.

Sol :

Dividing the polynomial x^3-3x^2+x+2 by a polynomial $g(x)$, the quotient $(x-2)$ and the remainder $(-2x+4)$ are obtained.

Dividend = Divisor × Quotient + Remainder

or$(x-2)\times g(x)+(-2x+4)$

$$=x^3-3x^2+x+2$$

$$(x-2)\times g(x)=x^3-3x^2+x+2+2x-4$$

$$g(x)=\frac{x^3-3x^2+3x-2}{x-2}$$

Now, dividing the polynomial x^3-3x^2+3x-2 by $x-2$

$$\begin{array}{r|l} & x^2-x+1 \\ x-2 & x^3-3x^2+3x-2 \\ & \underline{x^3-2x^2} \\ & \quad -x^2+3x-2 \\ & \quad \underline{-x^2+2x} \\ & \qquad\quad x-2 \\ & \qquad\quad \underline{x-2} \\ & \qquad\quad 0 \end{array}$$

Hence, $g(x)=x^2-x+1$ is obtained.

PRACTICE :

1. On dividing ax^4-4x^2+4 by a polynomial $g(x)$, the quotient and remainder were $3x^2-x$ and $-x+4$, respectively. Find $g(x)$.
   Ans : $3x^2+x-1$
2. On dividing $6x^3+11x^2-39x-65$ by a polynomial $g(x)$, the quotient and remainder were $6x+5$ and $-38x-60$ respectively. Find $g(x)$.
   Ans : x^2+x-1

5. Given example of polynomials $p(x)$, $g(x)$, $q(x)$ and $r(x)$, which satisfy the division algorithm and

(i) $\deg p(x) = \deg q(x)$

(ii) $\deg q(x) = \deg r(x)$

(iii) $\deg r(x) = 0$

Sol :

(i) $\deg p(x) = \deg q(x)$

We require $p(x)$ and $q(x)$ such that deg

$$p(x) = \deg q(x)$$

Then, $\quad \deg\ p(x) = \deg q(x)$

The degree of $g(x)$ must be zero.

Let, $\quad p(x) = 5x^2 - 5x + 10$

$$g(x) = 5$$

$$q(x) = x^2 - x + 2$$

$$r(x) = 0$$

$$\begin{array}{r|l} & x^2 - x + 2 \\ \hline 5 & 5x^2 - 5x + 10 \\ & 5x^2 \\ \hline & \quad -5x + 10 \\ & \quad -5x \\ \hline & \qquad 10 \\ & \qquad -10 \\ \hline & \qquad 0 \end{array}$$

By division algorithm,

$$5x^2 - 5x + 10 = 5(x^2 - x + 2) + 0$$

$$p(x) = g(x) \cdot q(x) + r(x)$$

Hence, $\quad \deg p(x) = \deg q(x) = 2$

(ii) $\deg q(x) = \deg r(x)$

We have $\quad \deg q(x) = \deg r(x)$

$$p(x) = g(x) \cdot q(x) + r(x)$$

Degree of $p(x)$ must be equal to the sum of degree of $g(x)$ and degree of $q(x)$.

Suppose $\quad p(x) - 7x^3 - 42x + 53$

$$g(x) = x^3 - 6x + 7$$

$$q(x) = 7$$

$$r(x) = 4$$

$$\begin{array}{r|l} & 7 \\ \hline x^3 - 6x + 7 & 7x^3 - 42x + 53 \\ & 7x^3 - 42x + 49 \\ \hline & \qquad 4 \end{array}$$

By division algorithm,

$$7x^3 - 42x + 53 = 7(x^3 - 6x + 7) + 4$$

So degree of $\quad q(x) = 0$

(iii) $\deg r(x) = 0$

We have $\quad \deg r(x) = 0$

Let, $\quad p(x) = x^3 + 2$

and $\quad g(x) = x^2 - x + 1$

Now dividing $x^3 + 2$ by $x^2 - x + 1$

$$\begin{array}{r|l} & x + 1 \\ \hline x^2 - x + 1 & x^3 + 2 \\ & x^3 - x^2 + x \\ \hline & x^2 - x + 2 \\ & x^2 - x + 1 \\ \hline & \qquad 1 \end{array}$$

By division algorithm

$$x^3 + 2 = (x^2 - x + 1)(x + 1) + 1$$

$$p(x) = g(x) \cdot q(x) + r(x)$$

Also $\quad \deg r(x) = 0$

PRACTICE :

1. Given example of polynomials $p(x)$, $g(x)$, $q(x)$ and $r(x)$, which satisfy the division algorithm and

(i) $\deg p(x) = \deg q(x)$

(ii) $\deg q(x) = \deg r(x)$

(iii) $\deg r(x) = 0$

Ans :

(i) $p(x) = 3x^2 - 3x + 12$, $q(x) = x^2 + x + 4$, $g(x) = 3$, $r(x) = 0$

(ii) $p(x) = 5x^3 - 10x + 26$, $g(x) = x^3 - 2x + 5$, $q(x) = 5$, $r(x) = 1$

(iii) $p(x) = 3x^2 - x^3 - 3x + 5$, $g(x) = x - 1 - x^2$, $q(x) = x - 2$, $r(x) = 3$

EXERCISE 1.4

1. Verify that the numbers given alongside of the cubic polynomials below are their zeroes. Also verify the relationship between the zeroes and the coefficients in each case :

(i) $2x^3 + x^2 - 5x + 2, \frac{1}{2}, 1, -2$

(ii) $x^3 - 4x^2 + 5x - 2;\ 2, 1, 1$

Sol :

(i) $2x^3 + x^2 - 5x + 2, \frac{1}{2}, 1, -2$

Comparing the given polynomial with the polynomial $ax^3 + bx^2 + cx + d$, we get

$$a = 2\ b = 1,\ c = -5 \text{ and } d = 2$$

Then, $\quad p\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 - 5\left(\frac{1}{2}\right) + 2$

$$= \frac{1}{4} + \frac{1}{4} - \frac{5}{2} + 2$$

$$= \frac{1 + 1 - 10 + 8}{4} = \frac{0}{4} = 0$$

$$p(1) = 2(1)^3 + (1)^2 + 5(1) + 2$$

$$= 2 + 1 - 5 + 2 = 0$$

$$\begin{aligned} p(-2) &= 2(-2)^3 + (-2)^2 - 5(-2) + 2 \\ &= 2(-8) + 4 + 10 + 2 \\ &= -16 + 16 = 0 \end{aligned}$$

Therefore, $\frac{1}{2}$, 1 and -2 are the zeroes of the given polynomial $2x^3 + x^2 - 5x + 2$ i.e.,

$$\alpha = \frac{1}{2},\ \beta = 1 \text{ and } \gamma = -2$$

Verification :

$$\begin{aligned} \alpha + \beta + \gamma &= \frac{1}{2} + 1 + (-2) \\ &= \frac{1+2-4}{2} = -\frac{1}{2} = \frac{-b}{a} \end{aligned}$$

$$\begin{aligned} \alpha\beta + \beta\gamma + \gamma\alpha &= \left(\frac{1}{2}\right)(1) + (1)(-2) + (-2)\left(\frac{1}{2}\right) \\ &= \frac{1}{2} - 2 - 1 = \frac{1-4-2}{2} \\ &= \frac{-5}{2} = \frac{c}{a} \end{aligned}$$

and
$$\begin{aligned} \alpha\beta\gamma &= \frac{1}{2} \times 1 \times -2 = -1 \\ &= \frac{-2}{2} = \frac{-d}{a} \end{aligned}$$

Hence verified. The relationship between the zeroes and the coefficient is correct.

(ii) $x^3 - 4x^2 + 5x - 2;\ 2,\ 1,\ 1$

Comparing the given polynomial with $ax^3 + bx^2 + cx + d$, $a = 1$, $b = -4$, $c = 5$ and $d = -2$

$$\begin{aligned} p(2) &= (2)^3 - 4(2)^2 + 5(2) - 2 \\ &= 8 - 16 + 10 - 2 = 0 \\ p(1) &= (1)^3 - 4(1)^2 + 5(1) - 2 \\ &= 1 - 4 + 5 - 2 = 0 \end{aligned}$$

2, 1 and 1 are the zeroes of the polynomial $x^3 - 4x^2 + 5x - 2$

So, $\alpha = 2$, $\beta = 1$ and $\gamma = 1$

Verification :

$$\begin{aligned} \alpha + \beta + \gamma &= 2 + 1 + 1 = 4 \\ &= \frac{-(-4)}{1} = \frac{-b}{a} \end{aligned}$$

$$\begin{aligned} \alpha\beta + \beta\gamma + \gamma\alpha &= (2)(1) + (1)(1) + (1)(2) \\ &= 2 + 1 + 2 = 5 = \frac{5}{1} = \frac{c}{a} \end{aligned}$$

and
$$\begin{aligned} \alpha\beta\gamma &= (2)(1)(1) = 2 \\ &= \frac{-(-2)}{1} = \frac{-d}{a} \end{aligned}$$

Hence verified. Therefore the above relationship of the zeroes of the polynomial with its coefficient is correct.

PRACTICE :

1. Verify that the numbers given alongside of the cubic polynomials below are their zeroes. Also verify the relationship between the zeroes and the coefficients in each case :
   (i) $x^3 - x^2 - 10x - 8;\ -1,\ -2,\ 4$
   (ii) $4x^3 - 11x^2 + 5x + 2;\ \frac{-1}{4},\ 1,\ 2$

2. Find a cubic polynomial with the sum, sum of the product of its zeroes taken two at a time, and the product of its zeroes as 2, -7, -14 respectively.

Sol :

Let a cubic polynomial be $ax^3 + bx^2 + cx + d$ and its zeroes are α, β and γ. Then

$$\alpha + \beta + \gamma = 2 = \frac{-(-2)}{1} = \frac{-b}{a}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = -7 = \frac{-7}{1} = \frac{c}{a}$$

and
$$\alpha\beta\gamma = -14 = \frac{-14}{1} = \frac{-d}{a}$$

If $a = 1$

Then, $b = -2$

$c = -7$

and $d = 14$

Thus the polynomial is $x^3 - 2x^2 - 7x + 14$

PRACTICE :

1. Find a cubic polynomial with the sum, sum of the product of its zeroes taken two at a time, and the product of its zeroes as 12, 39, 28 respectively.
   Ans : $x^3 - 12x^2 + 39x - 28$
2. Find a cubic polynomial with the sum, sum of the product of its zeroes taken two at a time, and the product of its zeroes as 3, 5, 3 respectively.
   Ans : $x^3 - 3x^2 + 5x - 3$

3. If the zeroes of the polynomial $x^3 - 3x^2 + x + 1$ are $a - b$, a, $a + b$ find a and b.

Sol :

$(a - b)$, a $(a + b)$ are the zeroes of the polynomial $x^3 - 3x^2 + x + 1$.

Therefore,

$$(a - b) + a + (a - b) = \frac{-(-3)}{1} = 3$$

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$3a = 3$$

$$a = 1$$

$$(a - b)a + a(a + b) + (a + b)(a - b) = \frac{1}{1} = 1$$

$$\alpha\beta + \beta r + r\alpha = \frac{c}{a}$$

$$a^2 - ab + a^2 + ab + a^2 - b^2 = 1$$

$$3a^2 - b^2 = 1$$

$$3(1)^2 - b^2 = 1 \qquad [\because a = 1]$$

$$3 - b^2 = 1$$

$$b^2 = 2 \Rightarrow b = \pm\sqrt{2}$$

Hence, $a = 1$ and $b = \pm\sqrt{2}$

PRACTICE :

1. If the zeroes of the polynomial $2x^3 - 15x^2 + 37x - 30$ are $a - b$, a, $a + b$ find a and b.

   Ans : $a = \frac{5}{2}$, $b = \pm\frac{1}{2}$

2. If the zeroes of the polynomial $x^3 - 6x^2 + 11x + 6$ are $a - b$, a, $a + b$ find a and b.

   Ans : $a = 2$, $b = \pm 1$

4. If two zeroes of the polynomial $x^4 - 6x^3 - 26x^2 + 138x - 35$ are $2 \pm \sqrt{3}$, find other zeroes.

Sol :

Two zeroes of the polynomial

$$p(x) = x^4 - 6x^3 - 26x^2 + 138x - 35$$

are $2 \pm \sqrt{3}$

Therefore, $x = 2 \pm \sqrt{3}$

$$x - 2 = \pm\sqrt{3}$$

Squaring both sides,

$$x^2 - 4x + 4 = 3$$

$$x^2 - 4x + 1 = 0$$

Now let us divide the polynomial $p(x)$ by $x^2 - 4x + 1$ so that other zeroes may be obtained.

$$\begin{array}{r|l} & x^2 - 2x - 35 \\ x^2 - 4x + 1 & x^4 - 6x^3 - 26x^2 + 138x - 35 \\ & x^4 - 4x^3 + x^2 \\ \hline & -2x^3 - 27x^2 + 138x - 35 \\ & -2x^3 + 8x^2 - 2x \\ \hline & -35x^2 + 140x - 35 \\ & -35x^2 + 140x - 35 \\ \hline & 0 \end{array}$$

$$\begin{aligned} p(x) &= x^4 - 6x^3 - 26x^2 + 138x - 35 \\ &= (x^2 - 4x + 1)(x^2 - 2x - 35) \\ &= (x^2 - 4x + 1)(x^2 - 7x + 5x - 35) \\ &= (x^2 - 4x + 1)[x(x - 7) + 5(x - 7)] \\ &= (x^2 - 4x + 1)(x + 5)(x - 7) \end{aligned}$$

$(x + 5)$ and $(x - 7)$ will be the other factors. Hence -5 and 7 will be the other zeroes.

PRACTICE :

1. If two zeroes of the polynomial $2x^4 + 7x^3 - 19x^2 - 14x + 30$ are $\sqrt{2}$, and $-\sqrt{2}$ find other zeroes.

   Ans : $-5, \frac{3}{2}$

2. If two zeroes of the polynomial $x^4 + x^3 - 34x^2 - 4x + 120$ are 2 and -2, find other zeroes.

   Ans : 5, -6

5. If the polynomial $x^4 - 6x^3 + 16x^2 - 25x + 10$ is divided by another polynomial $x^2 - 2x + k$, the remainder comes out to be $x + a$, find k and a.

Sol :

Dividing the polynomial $x^4 - 6x^3 + 16x^2 - 25x + 10$ by the polynomial $x^2 - 2x + k$

$$\begin{array}{r|l} & x^2 - 4x + (8 - k) \\ x^2 - 2x + k & x^4 - 6x^3 + 16x^2 - 25x + 10 \\ & x^4 - 2x^3 + kx^2 \\ \hline & -4x^3 + (16 - k)x^2 - 25x + 10 \\ & -4x^3 + 8x^2 - 4kx \\ \hline & (8 - k)x^2 + (4k - 25)x + 10 \\ & (8 - k)x^2 - 2(8 - k)x + (8 - k)k \\ \hline & (2k - 9)x - (8 - k)k + 10 \end{array}$$

$$\text{Remainder} = (2k - 9)x - (8 - k)k + 10$$

$$\text{But remainder} = x + a$$

Therefore, comparing the coefficients we have

$$2x - 9 = 1$$

$$2k = 10 \Rightarrow k = 5$$

and $-(8 - k)k + 10 = a$

$$\begin{aligned} a &= -(8 - 5)5 + 10 \\ &= -3 \times 5 + 10 = -15 + 10 \\ &= -5 \end{aligned}$$

Hence $k = 5$ and $a = -5$

PRACTICE :

1. If the polynomial $6x^4 + 8x^3 + 17x^2 + 21x + 7$ is divided by another polynomial $3x^2 + 4x + 1$, the remainder comes out to be $ax + b$, find a and b.

   Ans : $a = 1$, $b = 2$

2. If the polynomial $8x^4 + 14x^3 - 2x^2 + 7x - 8$ is divided by another polynomial $4x^2 + 3x + k$, the remainder comes out to $14x + a$, find $k$and a.

   Ans : $k = -2$, $a = -10$