2 Relations and Functions

Lesson at a Glance

1. A pair of elements grouped together in a particular order is called an **ordered pair.**
2. $(a, b) = (x, y)$ iff $a = x$ and $b = y$.
3. **Cartesian product** A × B of two sets A and B is defined as

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$
$$\text{and } n(A \times B) = n(A).\, n(B)$$
$$\text{i.e. if } n(A) = p \text{ and } n(B) = q,$$
$$\text{then } n(A \times B) = pq$$

4. $R \times R = \{(x, y) : x, y \in R\}$; $R \times R \times R = \{(x, y, z) : x, y, z \in R\}$.
5. $A \times \phi = \phi$, where ϕ is the empty set.
6. In general, $A \times B \neq B \times A$ but $n(A \times B) = n(B \times A)$ always, each $= n(A) \,.\, n(B)$.
7. A **relation** R from a set A to a set B is a subset of A × B. In fact, every subset of A × B is a relation from set A to set B.

   Therefore, number of relations from set A to set B

   = number of subsets of $A \times B = 2^{n(A \times B)} = 2^{n(A)n(B)}$.

   Every subset of A × A is called a relation from A → A or simply relation on the set A.
8. If $(x, y) \in R$, then y is called **image** of x under R and **Domain** of relation R is the set of all first elements of the ordered pairs in R.
9. **Range** of relation R is the set of all second elements of the ordered pairs in R.
10. **Function:** A relation R from A → B is called a function (*i.e.* $R \subset A \times B$) if **no two points of R have the same first entry** *i.e.* every element $x \in A$ has one and only one **image** $y \in B$.
11. If $f : A \to B$ is a function, then

Domain of f = A, co-domain of f = B.

Range of f = set $f(A) = \{f(x) : x \in A\}$. Range $\subset$ Co-domain.

12. If $f : A \to B$ is a function such that
    (*i*) range of $f = f(A) \subset R$, then f is called a **real valued function.**
    (*ii*) if $A \subset R$ and $f(A) \subset R$, then f is called a **real function.**
13. The function $f : R \to R$ defined by $f(x) = x$ for all $x \in R$ is called the **identity function.**
14. The function $f : R \to R$ defined by $f(x) = c$, $x \in R$ is called a **constant function.** Its range = $\{c\}$, a singleton set.
15. **Polynomial function.** A function of the form $f(x) = a_0x^n + a_1x^{n-1} + ... + a_n$ where n is a positive integer is called a polynomial function.
16. A **rational function** is of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain where $g(x) \neq 0$.
17. The function $f : R \to R$ defined by $f(x) = |x|$ for each $x \in R$ is called **modulus function.**

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$|x| \geq 0$ for all $x \in R$. $|x| = 0$ iff $x = 0$

If $x \neq 0$, then $|x| > 0$.

18. $|x|^2 = |x^2| = x^2$
19. $|-x| = |x|$ and $||x|| = |x|$
20. $|x| = \sqrt{x^2}$
21. The function $f : R \to R$ defined by

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases} \quad i.e., \ f(x) = \begin{cases} 1, & \text{if} \quad x > 0 \\ 0, & \text{if} \quad x = 0 \\ -1, & \text{if} \quad x < 0 \end{cases}$$

is called the **signum function.**

22. The function $f : R \to R$ defined by $f(x) = [x]$ is called the **greatest integer function.**

[x] = the greatest integer $\leq x$

If $x \in Z$, then $[x] = x$. If $x \notin Z$, then $[x]$ = the greatest integer $< x$.

23. If $f : X \to R$ and $g : X \to R$, then

(i) $(cf)(x) = c\, f(x),\ x \in X$.

(ii) $(f + g)(x) = f(x) + g(x),\ x \in X$.

(iii) $(f - g)(x) = f(x) - g(x),\ x \in X$.

(iv) $(fg)(x) = f(x)\, g(x),\ x \in X$.

(v) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(g)}{g(x)},\ x \in X,\ g(x) \neq 0.$

TEXTBOOK QUESTIONS SOLVED

EXERCISE 2.1 *(Page No.: 33-34)*

1. If $\left(\frac{x}{3}+1, y-\frac{2}{3}\right)=\left(\frac{5}{3},\frac{1}{3}\right)$, find the values of x and y.

Sol. Given: $\left(\frac{x}{3}+1, y-\frac{2}{3}\right)=\left(\frac{5}{3},\frac{1}{3}\right)$

Since the ordered pairs are equal, the corresponding elements are equal

$\therefore \quad \frac{x}{3} + 1 = \frac{5}{3}$ and $y - \frac{2}{3} = \frac{1}{3}$

$\Rightarrow \quad \frac{x}{3} = \frac{5}{3} - 1$ and $y = \frac{1}{3} + \frac{2}{3}$

$\Rightarrow \quad \frac{x}{3} = \frac{2}{3}$ and $y = \frac{3}{3}$

$\Rightarrow \quad x = 2$ and $y = 1$.

2. If the set A has 3 elements and the set B = { 3, 4, 5 }, then find the number of elements in (A × B).

Sol. Given: $n(A) = 3$ and $B = \{3, 4, 5\}$ so that $n(B) = 3$

$\therefore \quad n(A \times B) = n(A) \times n(B) = 3 \times 3 = 9.$

3. If G = { 7, 8 } and H = { 5, 4, 2 }, find G × H and H × G.

Sol. Given $G = \{7, 8\}$ and $H = \{5, 4, 2\}$

$\therefore \quad G \times H = \{(x, y) : x \in G \text{ and } y \in H\}$

$= \{(7, 5), (7, 4), (7, 2), (8, 5), (8, 4), (8, 2)\}$

and $H \times G = \{(x, y) : x \in H \text{ and } y \in G\}$

$= \{(5, 7), (5, 8), (4, 7), (4, 8), (2, 7), (2, 8)\}$.

4. State whether each of the following statements are true or false. If the statement is false, rewrite the given statement correctly.

(i) If $P = \{m, n\}$ and $Q = \{n, m\}$, then $P \times Q = \{(m, n), (n, m)\}$.

(ii) If A and B are non-empty sets, then $A \times B$ is a non-empty set of ordered pairs (x, y) such that $x \in A$ and $y \in B$.

(iii) If $A = \{1, 2\}$, $B = \{3, 4\}$, then $A \times (B \cap \phi) = \phi$.

Sol. (i) **False**

Since $n(P) = 2$ and $n(Q) = 2$, therefore,

$n(P \times Q) = n(P) . n(Q) = 2 . 2 = 4$.

Correct statement is

$P \times Q = \{(m, n), (m, m), (n, n), (n, m)\}$.

(ii) **True.**

(iii) $A \times (B \cap \phi) = A \times \phi = A \times \{\ \} = \phi$

$\therefore$ The given statement is true.

5. If $A = \{-1, 1\}$, find $A \times A \times A$.

Sol. Given: $A = \{-1, 1\}$

$\Rightarrow$ $A \times A = \{-1, 1\} \times \{-1, 1\}$

$= \{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$

$\therefore$ $A \times A \times A = \{(-1, -1, -1), (-1, -1, 1), (-1, 1, -1), (-1, 1, 1), (1, -1, -1), (1, -1, 1), (1, 1, -1), (1, 1, 1)\}$.

6. If $A \times B = \{(a, x), (a, y), (b, x), (b, y)\}$, find A and B.

Sol. A = set of first elements

$= \{a, b\}$ [Dropping repetitions]

B = set of second elements

$= \{x, y\}$. [Dropping repetitions]

7. Let $A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$, $C = \{5, 6\}$ and $D = \{5, 6, 7, 8\}$. Verify that

(i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

(ii) $A \times C$ is a subset of $B \times D$.

Sol. (i) $B \cap C = \{1, 2, 3, 4\} \cap \{5, 6\} = \phi$

$\therefore$ $A \times (B \cap C) = A \times \phi = \phi$ ...(i)

Also $A \times B = \{1, 2\} \times \{1, 2, 3, 4\}$

$= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4)\}$

and $A \times C = \{1, 2\} \times \{5, 6\}$

$= \{(1, 5), (1, 6), (2, 5), (2, 6)\}$

$\therefore$ $(A \times B) \cap (A \times C) = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4)\} \cap \{(1, 5), (1, 6), (2, 5), (2, 6)\}$

$= \phi$ ...(*ii*)

From (*i*) and (*ii*),

$A \times (B \cap C) = (A \times B) \cap (A \times C)$ is verified.

(*ii*) $A \times C = \{1, 2\} \times \{5, 6\}$

$= \{(1, 5), (1, 6), (2, 5), (2, 6)\}$

$B \times D = \{1, 2, 3, 4\} \times \{5, 6, 7, 8\}$

$= \{(1, 5), (1, 6), (1, 7), (1, 8), (2, 5), (2, 6), (2, 7), (2, 8), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8)\}$

Every element of $A \times C$ is also an element of $B \times D$.

$\therefore$ $A \times C \subset B \times D$ is verified.

8. Let A = { 1, 2 } and B = { 3, 4 }. Write A × B. How many subsets will A × B have? List them.

Sol. Here, $A = \{1, 2\}$ and $B = \{3, 4\}$

$\therefore$ $A \times B = \{(a, b) : a \in A, b \in B\}$

$= \{(1, 3), (1, 4), (2, 3), (2, 4)\} = \{a, b, c, d\}$ (say)

Since $n(A \times B) = 4$, therefore, $A \times B$ will have $2^4 = 16$ subsets.

The subsets of $A \times B$ are

$\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}$

Putting values of $a = (1, 3), b = (1, 4), c = (2, 3), d = (2, 4)$; these subsets are

$\phi, \{(1,3)\}, \{(1,4)\}, \{(2,3)\}, \{(2,4)\}, \{(1,3),(1,4)\}, \{(1,3),(2,3)\}, \{(1,3),(2,4)\}, \{(1,4),(2,3)\}, \{(1,4),(2,4)\}, \{(2,3),(2,4)\}, \{(1,3),(1,4),(2,3)\}, \{(1,3),(1,4),(2,4)\}, \{(1,3),(2,3),(2,4)\}, \{(1,4),(2,3),(2,4)\}, \{(1,3),(1,4),(2,3),(2,4)\}$.

9. Let A and B be two sets such that *n*(A) = 3 and *n*(B) = 2. If (*x*, 1), (*y*, 2), (*z*, 1) are in A × B, find A and B, where *x*, *y* and *z* are distinct elements.

Sol. Since $n(A) = 3$ and $n(B) = 2$;

therefore,

$n(A \times B) = n(A) \, . \, n(B) = 3 \, . \, 2 = 6$

We know that elements of set A are the first elements and elements of set B are the second elements in the ordered pairs of A × B. Also $(x, 1)$, $(y, 2)$, $(z, 1)$ are in A × B.

∴ Set A = $\{x, y, z\}$ and set B = {1, 2}.

10. The Cartesian product A × A has 9 elements among which are found (– 1, 0) and (0, 1). Find the set A and the remaining elements of A × A.

Sol. Since A × A has nine elements, *i.e.*, $n(A \times A) = 9 = 3 \times 3$; therefore, $n(A)$ must be 3.

Now two elements of A × A are (–1, 0) and (0, 1) (given)

Therefore, set A = {–1, 0, 1} because in A × A, elements at both places of ordered pairs are from A.

$$\therefore \quad A \times A = \{-1, 0, 1\} \times \{-1, 0, 1\}$$
$$= \{(-1, -1), (-1, 0), (-1, 1), (0, -1), (0, 0), (0, 1), (1, -1), (1, 0), (1, 1)\}$$

∴ The remaining elements of A × A other than the given two elements (– 1, 0) and (0, 1) of A × A are (– 1, – 1), (– 1, 1), (0, – 1), (0, 0), (1, – 1), (1, 0), (1, 1).

EXERCISE 2.2 *(Page No.: 35-36)*

1. Let A = {1, 2, 3, ..., 14}. Define a relation R from A to A by R = $\{(x, y) : 3x - y = 0$, where $x, y \in A\}$. Write down its domain, codomain and range.

Sol. Here, A = {1, 2, 3, ..., 14} and

R = $\{(x, y) : 3x - y = 0$, where $x, y \in A\}$

= $\{(x, y) : y = 3x$, where $x = 1, 2, 3, 4\}$

(when $x \geq 5$, $y = 3x \geq 15$, so that $y \notin A$)

= {(1, 3), (2, 6), (3, 9), (4, 12)}

Domain of R = the set of first elements of the ordered pairs in R

= {1, 2, 3, 4}

Codomain of R = A = {1, 2, 3, ..., 14}

Range of R = the set of second elements of the ordered pairs in R

= {3, 6, 9, 12}.

2. Define a relation R on the set N of natural numbers by R = { $(x, y) : y = x + 5$, x is a natural number less than 4; $x, y \in$ N }. Depict this relationship using roster form. Write down the domain and the range.

Sol. Here, R = { $(x, y) : y = x + 5, x \in$ N and $x < 4$ }

= { $(x, y) : y = x + 5$ and $x = 1, 2, 3$ }

= { (1, 1 + 5), (2, 2 + 5), (3, 3 + 5) }

= { (1, 6), (2, 7), (3, 8) }

Domain of R = { 1, 2, 3 }

Range of R = { 6, 7, 8 }.

3. A = { 1, 2, 3, 5 } and B = { 4, 6, 9 }. Define a relation R from A to B by R = { (x, y) : the difference between x and y is odd; $x \in$ A, $y \in$ B }. Write R in roster form.

Sol. Here, R = { (x, y) : the difference between x and y is odd; $x \in$ A, $y \in$ B }

= { $(x, y) : |x - y|$ is odd; $x \in$ A, $y \in$ B }

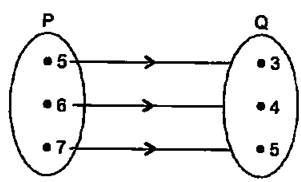
= { (1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5 4), (5, 6) }.

4. The Fig. shows a relationship between the sets P and Q. Write this relation

(*i*) in set-builder form

(*ii*) in roster form.

What is its domain and range?



Sol. Let R be the relation from P to Q.

Clearly from the fig., the ordered pairs (5, 3), (6, 4) and (7, 5) belong to R. The second element of each ordered pair is two less than the corresponding first element.

Therefore,

(*i*) In set-builder form

R = { $(x, y) : y = x - 2$, where $x = 5, 6, 7$ }

(*ii*) In roster form

R = { (5, 3), (6, 4), (7, 5) }

Domain of R = { 5, 6, 7 }

Range of R = { 3, 4, 5 }.

5. Let A = { 1, 2, 3, 4, 6 }. Let R be the relation on A defined by { $(a, b) : a, b \in$ A, b is exactly divisible by a }.

(*i*) Write R in roster form (*ii*) Find the domain of R (*iii*) Find the range of R.

Sol. Here, A = { 1, 2, 3, 4, 6 } and

(*i*) R = { (a, b) : $a, b \in$ A, b is exactly divisible by a}
= { (1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6) }

(*ii*) Domain of R = { 1, 2, 3, 4, 6 } = A

(*iii*) Range of R = { 1, 2, 3, 4, 6 } = A.

6. Determine the domain and range of the relation R defined by R = { $(x, x + 5)$: $x \in$ { 0, 1, 2, 3, 4, 5 }}.

Sol. R = { $(x, x + 5)$: $x \in$ { 0, 1, 2, 3, 4, 5 }}
= { (0, 0 + 5), (1, 1 + 5), (2, 2 + 5), (3, 3 + 5), (4, 4 + 5), (5, 5 + 5) }
= { (0, 5), (1, 6)), (2, 7), (3, 8), (4, 9), (5, 10) }

Domain of R = { 0, 1, 2, 3, 4, 5 }

Range of R = { 5, 6, 7, 8, 9, 10 }.

7. Write the relation R = { (x, x^3) : x is a prime number less than 10 } in roster form.

Sol. Prime numbers less than 10 are 2, 3, 5, 7

$\therefore$ x = 2, 3, 5, 7

R = { (x, x^3) : x = 2, 3, 5, 7 }
= { $(2, 2^3)$, $(3, 3^3)$, $(5, 5^3)$, $(7, 7^3)$ }
= { (2, 8), (3, 27), (5, 125), (7, 343) }.

8. Let A = { x, y, z } and B = { 1, 2 }. Find the number of relations from A to B.

Sol. Given: A = { x, y, z } and B = { 1, 2 }

$\Rightarrow$ $n(A) = 3$ and $n(B) = 2$

so that $n(A \times B) = n(A) \cdot n(B) = 3 \cdot 2 = 6$

Number of relations from A to B
= Number of subsets of A × B
= $2^6 = 64$.

9. Let R be the relation on Z defined by R = {(a, b) : $a, b \in$ Z, $a - b$ is an integer}. Find the domain and range of R.

Sol. R = { (a, b) : $a, b \in$ Z, $a - b$ is an integer }

Since $a - b$ is an integer for all $a, b \in$ Z

$\therefore$ R = { (a, b) : $a, b \in$ Z }

$\Rightarrow$ Domain of R = Z

Range of R = Z.

EXERCISE 2.3 *(Page No.: 44)*

1. Which of the following relations are functions? Give reasons. If it is a function, determine its domain and range.

(i) { (2, 1), (5, 1), (8, 1), (11, 1), (14, 1), (17, 1) }

(ii) { (2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7) }

(iii) { (1, 3), (1, 5), (2, 5) }.

Sol. (i) Domain of the relation = { 2, 5, 8, 11, 14, 17 }

Since every element of the domain has a unique image, **this relation is a function.**

Domain of the function = { 2, 5, 8, 11, 14, 17 }

Range of the function = { 1 }.

(ii) Domain of the relation = { 2, 4, 6, 8, 10, 12, 14 }

Since every element of the domain has a unique image, **this relation is a function.**

Domain of the function = { 2, 4, 6, 8, 10, 12, 14 }

Range of the function = { 1, 2, 3, 4, 5, 6, 7 }.

(iii) Domain of the relation = { 1, 2 }

Since the same first element 1 corresponds to two different images 3 and 5, **this relation is not a function.**

Note: Method to find domain and range of the function when rule of the function $f(x)$ is given.

Domain of a function $f(x)$ is the set of all those **real numbers** for which $f(x)$ is real and finite.

i.e. some $x \in R$ does not belong to the domain of a function $f(x)$ if $f(x) \to \infty$ for this value of x or $f(x)$ assumes a value which is square root of a negative real number.

Case I If $f(x) = \dfrac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are polynomial function of x; then the points where $h(x)=0$ do not belong to the domain of $f(x)$.

Case II If $f(x) = \sqrt{g(x)}$, then only those points where $g(x) \geq 0$ belong to the domain of f.

Case III If $f(x) = \dfrac{1}{\sqrt{g(x)}}$, then only those points where $g(x) > 0$ belong to the domain of f.

Range of a function is the set of values of $y = f(x)$ for values of x in the domain.

2. Find the domain and range of the following real functions:

(*i*) $f(x) = -|x|$ (*ii*) $f(x) = \sqrt{9-x^2}$.

Sol. (*i*) Since $|x|$ is defined *i.e.* is real and finite for all real numbers, $-|x|$ is also defined for all real numbers.

$\therefore$ Domain of f = R

Also, for all $x \in$ R, $|x| \geq 0$

$\Rightarrow -|x| \leq 0 \qquad \Rightarrow f(x) \leq 0$

$\therefore$ Range of $f = (-\infty, 0]$.

(*ii*) $f(x) = \sqrt{9-x^2} \qquad \Rightarrow y = \sqrt{9-x^2}$

$f(x)$ is defined *i.e.* real and finite if $9 - x^2 \geq 0$

$\Rightarrow 3^2 - x^2 \geq 0 \Rightarrow (3-x)(3+x) \geq 0$

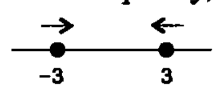
Case I $3 - x \geq 0$ and $3 + x \geq 0$ $\qquad (\because (+)(+) = +)$

$\Rightarrow -x \geq -3$ and $x \geq -3$

multiplying by (-1), (Multiplication by a negative number reverses the inequality)

$\Rightarrow x \leq 3$ and $x \geq -3$

$\therefore \quad x \in [-3, 3]$ ... (*i*)

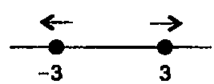


Case II $3 - x \leq 0$ and $3 + x \leq 0$ $(\because (-)(-)$ is also $+)$

$\Rightarrow -x \leq -3$ and $x \leq -3$

$\Rightarrow x \geq 3$ and $x \leq -3$

But this is impossible



$\therefore$ From (*i*), Domain of $f(x)$ is $[-3, 3]$

To find range, we should (have to) find x in terms of y.

$$y = \sqrt{9-x^2} \quad \Rightarrow \quad y^2 = 9 - x^2$$

$$\Rightarrow \qquad x^2 = 9 - y^2$$

$$\Rightarrow \qquad x = \pm\sqrt{9-y^2}$$

Now x is real if $9 - y^2 \geq 0$

$\Rightarrow (3 - y)(3 + y) \leq 0$

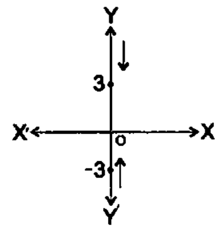
Case I $3 - y \geq 0$ and $3 + y \geq 0$

$\Rightarrow -y \geq -3$ and $y \geq -3$

$y \leq 3$ and $y \geq -3$

$\therefore -3 \leq y \leq 3$

$\therefore y \in [-3, 3]$ ...(*ii*)

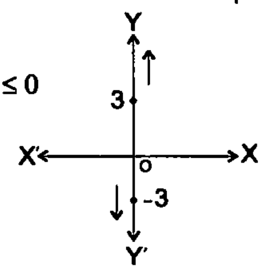


Case II $3 - y \leq 0$ and $3 + y \leq 0$

$\Rightarrow -y \leq -3$ and $y \leq -3$

$\Rightarrow y \geq 3$ and $y \leq -3$

But this is impossible.



But $y = \sqrt{9 - x^2}$ **is always** ≥ 0 ...(*iii*)

From (*ii*) and (*iii*), $0 \leq y \leq 3$

$\therefore$ Range (set) is closed interval [0, 3].

3. A function f is defined by $f(x) = 2x - 5$. Write down the values of

(*i*) $f(0)$ (*ii*) $f(7)$ (*iii*) $f(-3)$.

Sol. Given: $f(x) = 2x - 5$

(*i*) Replacing x by 0, we have

$$f(0) = 2 \times 0 - 5 = -5$$

(*ii*) Replacing x by 7, we have

$$f(7) = 2 \times 7 - 5 = 14 - 5 = 9$$

(*iii*) Replacing x by -3, we have

$$f(-3) = 2 \times (-3) - 5 = -6 - 5 = -11.$$

4. The function 't' which maps temperature in degree Celsius into temperature in degree Fahrenheit is defined by $t(C) = \frac{9C}{5} + 32$. Find

(*i*) $t(0)$ (*ii*) $t(28)$ (*iii*) $t(-10)$

(*iv*) The value of C, when $t(C) = 212$.

Sol. Given: $t(C) = \frac{9C}{5} + 32$

(*i*) Replacing C by 0, we have

$$t(0) = \frac{9 \times 0}{5} + 32 = 0 + 32 = 32$$

(ii) Replacing C by 28, we have

$$t(28) = \frac{9 \times 28}{5} + 32 = \frac{252}{5} + 32$$
$$= 50.4 + 32 = 82.4$$

(iii) Replacing C by − 10, we have

$$t(-10) = \frac{9 \times (-10)}{5} + 32 = -18 + 32 = 14.$$

(iv) $t(C) = 212 \quad \Rightarrow \quad \frac{9C}{5} + 32 = 212$

$$\Rightarrow \quad \frac{9C}{5} = 212 - 32 = 180 \quad \Rightarrow \quad C = \frac{5}{9} \times 180 = 100.$$

5. Find the range of each of the following functions:

(i) $f(x) = 2 - 3x, x \in R, x > 0$.

(ii) $f(x) = x^2 + 2$, x is a real number.

(iii) $f(x) = x$, x is a real number.

Sol. (i) $f(x) = 2 - 3x, x \in R, \quad x > 0$ ...(i) (given)

Consider $x > 0$ (given)

Multiplying both sides by − 3, $-3x < 0$

Adding 2 to both sides, $2 - 3x < 2$

or (By (i)) $f(x) < 2$ for $x > 0$

$\Rightarrow f(x) \in (-\infty, 2) \Rightarrow$ Range of this $f(x)$ is $(-\infty, 2)$.

Note. Domain of this $f(x)$ is $\{x : x \in R, x > 0\}$ (given) $= (0, \infty)$

(ii) $f(x) = x^2 + 2, x \in R$ ...(i) (given)

$$x \in R \Rightarrow x^2 \geq 0$$

Adding 2 to both sides, $x^2 + 2 \geq 2$

or (By (i)) $f(x) \geq 2 \Rightarrow f(x) \in [2, \infty)$

$\therefore$ Range of $f(x)$ is $[2, \infty)$

(iii) $f(x) = x; x \in R$ (given)

$\Rightarrow f(x)$ also takes all real values.

$\therefore$ Range of $f(x)$ is R.

MISCELLANEOUS EXERCISE ON CHAPTER 2

(*Page No.: 46 - 47*)

1. The relation f is defined by $f(x) = \begin{cases} x^2, & 0 \le x \le 3 \\ 3x, & 3 \le x \le 10 \end{cases}$

The relation g is defined by $g(x) = \begin{cases} x^2, & 0 \le x \le 2 \\ 3x, & 2 \le x \le 10 \end{cases}$.

Show that f is a function and g is not a function.

Sol. The domain of f is [0, 10].

For each $x \in [0, 3)$, $f(x) = x^2$ is uniquely defined

i.e. each $x \in [0, 3]$ has a unique image

For each $x \in (3, 10]$, $f(x) = 3x$ is uniquely defined.

At $x = 3$, $x^2 = 3^2 = 9$. Also $3x = 3 \times 3 = 9$ so that f is uniquely defined at 3.

Since every point of the domain has one and only one image under f, this relation is a function.

The domain of g is [0, 10].

For each $x \in [0, 2)$, $f(x) = x^2$ is uniquely defined.

For each $x \in (2, 10]$, $f(x) = 3x$ is uniquely defined.

At $x = 2$, $g(x) = x^2 = 2^2 = 4$. Also at $x = 2$, $g(x) = 3x = 3 \times 2 = 6$.

Now, 2 is in the domain of g and the image of 2 under g is not unique since $(2, 4), (2, 6) \in g$, therefore, g is not a function.

2. If $f(x) = x^2$, find $\dfrac{f(1.1) - f(1)}{1.1 - 1}$.

Sol. $$\frac{f(1.1) - f(1)}{1.1 - 1} = \frac{(1.1)^2 - (1)^2}{1.1 - 1} = \frac{(1.1 + 1)(1.1 - 1)}{1.1 - 1}$$
$$= 1.1 + 1 = 2.1.$$

3. Find the domain of the function $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - 8x + 12}$.

Sol. f is a rational function.

$\therefore$ Domain of $f = R - \{x : x^2 - 8x + 12 = 0\}$

(See case I, **Note** after solution of Q.N.1, page 48)

$= R - \{x : (x - 2)(x - 6) = 0\}$

$= R - \{2, 6\}$.

4. Find the domain and the range of the real function f defined by $f(x) = \sqrt{x-1}$.

Sol. f is real only when $x - 1 \geq 0$ *i.e.,* $x \geq 1$

(See case II, **Note** after solution of Q.N.1, page 48)

$\therefore$ Domain of $f = [1, \infty)$

Let $y = f(x) = \sqrt{x-1}$

Clearly, $y \geq 0$ ...(*i*)

Also, $y^2 = x - 1$ or $x = 1 + y^2$

x is real for all $y \in R$ ...(*ii*)

Range of f is the set of values common to (*i*) and (*ii*)

$\therefore$ Range of $f = \{y : y \geq 0, y \in R\}$

$= [0, \infty)$.

5. Find the domain and the range of the real function f defined by

$$f(x) = |x - 1|.$$

Sol. For all $x \in R$, $f(x)$ is a unique real number.

$\therefore$ Domain of $f = R$.

Also, for all $x \in R$, $|x - 1| \geq 0$

$\Rightarrow f(x) \geq 0$

$\therefore$ Range of $f = [0, \infty)$.

6. Let $f = \left\{\left(x, \dfrac{x^2}{1+x^2}\right) : x \in R\right\}$ be a function from R into R. Determine the range of f.

Sol. $y = (f(x)) = \dfrac{x^2}{1+x^2}$ (given) ...(*i*)

To find range of f

From (*i*), $y = \dfrac{x^2}{1+x^2}$ is ≥ 0 for all $x \in R$ $(\because x^2 \geq 0)$ ...(*ii*)

Again from (*i*), $y = \dfrac{x^2}{1+x^2} = \dfrac{1+x^2-1}{1+x^2}$

$$= \frac{1+x^2}{1+x^2} - \frac{1}{1+x^2} = 1 - \frac{1}{1+x^2} < 1 \ \forall\ x \in R \quad ...(iii)$$

From (*ii*) and (*iii*), $y \geq 0$ and $y < 1$

$\therefore \quad 0 \leq y = f(x) < 1$

$\therefore$ Range of $f(x)$ is $[0, 1)$.

7. Let $f, g : \mathbf{R} \to \mathbf{R}$ be defined, respectively by

$f(x) = x + 1$, $g(x) = 2x - 3$. Find $f + g$, $f - g$ and $\frac{f}{g}$.

Sol. $(f + g)(x) = f(x) + g(x) = (x + 1) + (2x - 3) = 3x - 2$

$(f - g)(x) = f(x) - g(x) = (x + 1) - (2x - 3) = -x + 4$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0 \quad = \frac{x+1}{2x-3}, \quad 2x - 3 \neq 0$$

$$= \frac{x+1}{2x-3}, \quad x \neq \frac{3}{2}.$$

8. Let $f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}$ be a function form Z to Z defined by $f(x) = ax + b$, for some integers a, b. Determine a, b.

Sol. Given: $f(x) = ax + b$ ...(*i*)

Now, $(1, 1) \in f \Rightarrow f(1) = 1$

Replacing x by 1 in (*i*), we have

$f(1) = a \times 1 + b \quad \Rightarrow \quad 1 = a + b$ ...(*ii*)

Also $(2, 3) \in f \quad \Rightarrow \quad f(2) = 3$

Again, replacing x by 2 in (*i*), we have

$f(2) = a \times 2 + b \quad \Rightarrow \quad 3 = 2a + b$ ...(*iii*)

Subtracting (*ii*) from (*iii*), we get $2 = a$

Putting this value of a in (*ii*), we get

$1 = 2 + b$ or $\quad b = -1$

$\therefore \quad a = 2, b = -1.$

9. Let R be a relation from N to N defined by R = $\{(a, b) : a, b \in \mathbf{N}$ and $a = b^2\}$. Are the following true?

(*i*) $(a, a) \in \mathbf{R}$, for all $a \in \mathbf{N}$

(*ii*) $(a, b) \in \mathbf{R}$, implies $(b, a) \in \mathbf{R}$

(*iii*) $(a, b) \in \mathbf{R}$, $(b, c) \in \mathbf{R}$ implies $(a, c) \in \mathbf{R}$

Justify your answer in each case.

Sol. (*i*) We know that except natural number 1, none of the natural numbers is equal to its square.

$\therefore$ $(a, a) \notin$ R (It does not satisfy the condition $a = b^2$)

$\therefore$ $(a, a) \in$ R is false.

(*ii*) If $a = b^2$, it does not imply that $b = a^2$

i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ is not true.

For example, $(4, 2) \in R$

$(\because\ a = 4, b = 2$ and $a = b^2)$ but $(2, 4) \notin R$ $(\because\ 2 \neq 4^2)$

(*iii*) If $(a, b) \in R \Rightarrow a = b^2$ ..(*i*)

also $(b, c) \in R \Rightarrow b = c^2$ ..(*ii*)

Putting the value of b from (*ii*) in (*i*),

$$a = c^4$$

$$\therefore\quad a \neq c^2 \Rightarrow (a, c) \notin R$$

For example, $(a, b) = (16, 4) \in R$ and $(b, c) = (4, 2) \in R$

$(\because$ For each pair $a = b^2)$

but $(a, c) = (16, 2) \notin R$ $(\because\ 16 \neq 2^2)$

$\therefore$ The statement is false.

10. Let A = { 1, 2, 3, 4 }, B = { 1, 5, 9, 11, 15, 16 } and *f* = { (1, 5), (2, 9), (3, 1), (4, 5), (2, 11) }. Are the following true?

(*i*) *f* is a relation from A to B

(*ii*) *f* is a function from A to B

Justify your answer in each case.

Sol. (*i*) Every element of f is an element of $A \times B$

$\Rightarrow f \subset A \times B$

$\Rightarrow f$ is a relation from A to B (since every subset of $A \times B$ is a relation from A to B).

(*ii*) (2, 9) and (2, 11) both belong to f

Here both these points of f have the same first entry

$\Rightarrow f$ - image of 2 is not unique

$\Rightarrow f$ is not a function from A to B.

11. Let *f* be the subset of Z × Z defined by $f = \{ (ab, a + b) : a, b \in Z \}$. Is *f* a function from Z to Z? Justify your answer.

Sol. Given: $f = \{ (ab, a + b) : a, b \in Z \}$

Taking $a = b = 1$, we have $(ab, a + b) = (1, 2) \in f$

Taking $a = b = -1$, we have $(ab, a + b) = (1, -2) \in f$

$\Rightarrow f$ - image of 1 is not unique

$\Rightarrow f$ is not a function.

12. Let A = { 9, 10, 11, 12, 13 } and let $f : A \to N$ be defined by $f(n)$ = the highest prime factor of n. Find the range of *f*.

Sol. Here, A = {9, 10, 11, 12, 13} and $f(n)$ = the highest prime factor of n.

The only prime factor of 9 is 3.

$\Rightarrow \quad f(9) = 3$

Prime factors of 10 are 2 and 5.

$\Rightarrow \quad f(10) = 5$

The only prime factor of 11 is 11

$\Rightarrow \quad f(11) = 11$

Prime factors of 12 are 2 and 3

$\Rightarrow \quad f(12) = 3$

The only prime factor of 13 is 13

$\Rightarrow \quad f(13) = 13$

$\therefore$ Range of f = set of f-images of elements in the domain A

$= \{f(9), f(10), f(11), f(12), f(13)\}$

$= \{3, 5, 11, 3, 13\}$

$= \{3, 5, 11, 13\}$. [Dropping repetitions]