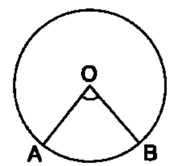
3 Trigonometric Functions

Lesson at a Glance

1. Definition: **Radian.** Radian is an angle subtended at the centre of the circle by an arc length equal to the radius of the circle. In the adjoining figure,



$\angle$ AOB = 1 Radian where arc AB = Radius of the circle and O is the centre of the circle.

2. In a circle of radius r, if an arc of length l subtends an angle θ radians at the centre, then $l = r\theta$.

3. (*a*) Radian measure = Degree measure $\times \dfrac{\pi}{180^\circ}$.

($\because$ $180^\circ = \pi$ Radians).

(*b*) Degree measure = Radian measure $\times \dfrac{180^\circ}{\pi}$.

($\because$ π Radians = 180°)

(*c*) 1° = 60 minutes (= 60') and 1 minute = 60 seconds (= 60")

4. (*a*) $\operatorname{cosec} x = \dfrac{1}{\sin x}$ (*b*) $\sec x = \dfrac{1}{\cos x}$ (*c*) $\cot x = \dfrac{1}{\tan x}$.

5. (*a*) $\tan x = \dfrac{\sin x}{\cos x}$ (*b*) $\cot x = \dfrac{\cos x}{\sin x}$.

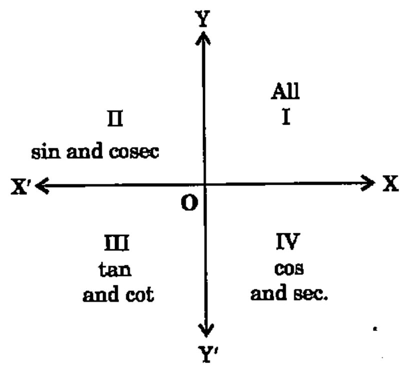
6. (*i*) $\cos^2 x + \sin^2 x = 1$.
   (*ii*) $\sec^2 x - \tan^2 x = 1$.
   (*iii*) $\operatorname{cosec}^2 x - \cot^2 x = 1$.
   ] square relations

7. (*i*) $-1 \le \sin x \le 1$ and $-1 \le \cos x \le 1$ i.e. $\sin x$ and $\cos x \in [-1, 1]$.
   (*ii*) $\sec x \le -1$ or ≥ 1 and $\operatorname{cosec} x \le -1$ or ≥ 1
   i.e. $\sec x$ and $\operatorname{cosec} x \in R - (-1, 1)$
   (*iii*) tax x and cot x can assume all real values

8. (*i*) Period of $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ is 2π

(*ii*) Period of $\tan x$ and $\cot x$ is π.

9. **Rule. ASTC**



10. **Three important rules for *t*-ratios of allied angles:**

Rule 1. ***T*-ratios of $(2n\pi + \theta)$ or $(n \times 360° + \theta)$ are the same as those of θ, *n* being an integer.**

For example, $\sin(6\pi + \theta) = \sin(3 \times 2\pi + \theta) = \sin\theta$,

$\cos(8 \times 360° + \theta) = \cos\theta$ and $\tan(5 \times 360° + 45°) = \tan 45° = 1$.

Thus, any number of revolutions, *i.e.*, $n \times 2\pi$, $n \in Z$ can be **added to or taken away from an angle** without affecting its *t*-ratios.

Rule 2. ***To write down the t-ratios of an angle allied to θ.***

(*i*) Assume θ to be a positive acute angle (even if not so, because the form of the result is the same for all values of θ) and determine the quadrant in which the allied angle lies and

Use the rule 'All-sin-tan-cos' to find the sign of the *t*-ratios in this quadrant.

Thus,

$90° - \theta$, $360° + \theta$ lie in 1st quadrant.

$90° + \theta$, $180° - \theta$ lie in 2nd quadrant.

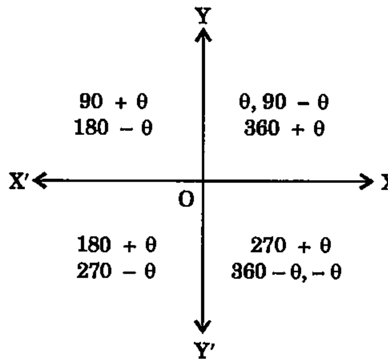
$180° + \theta$, $270° - \theta$ lie in 3rd quadrant.

$-\theta$, $360° - \theta$, $270° + \theta$ lie in 4th quadrant.

The quadrant of the above angles can easily be learnt from the given figure on next page.

(*ii*) If the allied angle is $-\theta$, $180° \pm \theta$, $360° \pm \theta$, etc. (*i.e.*, $n \times 90° \pm \theta$ where n is an **even** integer), the ***t*-ratio is not altered.**

(*iii*) If the allied angle is $90° \pm \theta$, $270° \pm \theta$, etc. (*i.e.*, $n \times 90° \pm \theta$) where n is an **odd** integer, the *t*-ratio changes.



Thus, sin $\rightleftharpoons$ cos,

tan $\rightleftharpoons$ cot, sec $\rightleftharpoons$ cosec.

Hence, add 'co' if absent and remove 'co' if present.

Remark. While applying Rule 2,

Firstly apply rule 2 (*i*) namely All-sin-tan-cos and then apply rule 2 (*ii*) or rule 2 (*iii*).

Rule 3. ***To express t-ratios of any negative angle in terms of those of a positive acute angle.***

If the angle is negative, make it positive by using the formulae for *t*-ratios of $-\theta$ (of course we know from the above figure that $-\theta$ lies in 4th quadrant).

$\cos(-\theta) = \cos\theta$, $\sec(-\theta) = \sec\theta$, $\sin(-\theta) = -\sin\theta$, $\text{cosec}(-\theta) = -\text{cosec}\,\theta$, $\tan(-\theta) = -\tan\theta$, $\cot(-\theta) = -\cot\theta$.

Now apply Rules I and II

11. (*a*) $\sin(x + y) = \sin x \cos y + \cos x \sin y$

(*b*) $\sin(x - y) = \sin x \cos y - \cos x \sin y$.

12. (*a*) $\cos(x + y) = \cos x \cos y - \sin x \sin y$

(*b*) $\cos(x - y) = \cos x \cos y + \sin x \sin y$.

13. (*a*) $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

(*b*) $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$.

(*c*) $\cot(A + B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$

(*d*) $\cot(A - B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

14. (*a*) $\tan\left(\dfrac{\pi}{4} + x\right) = \dfrac{1 + \tan x}{1 - \tan x}$

(*b*) $\tan\left(\dfrac{\pi}{4} - x\right) = \dfrac{1 - \tan x}{1 + \tan x}$.

15. $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$.

16. $\cos(x + y)\cos(x - y) = \cos^2 x - \sin^2 y$.

17. $2\sin A\cos B = \sin(A + B) + \sin(A - B)$.

18. $2\cos A\sin B = \sin(A + B) - \sin(A - B)$.

19. $2\cos A\cos B = \cos(A + B) + \cos(A - B)$.

20. $2\sin A\sin B = \cos(A - B) - \cos(A + B)$.

(C – D Formulae)

21. $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$.

22. $\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$.

23. $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$.

24. $\cos C - \cos D = -2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$.

$= 2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}$.

25. $\sin 2x = 2\sin x\cos x = \dfrac{2\tan x}{1 + \tan^2 x}$.

26. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$= \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$.

27. From result 26; $\sin^2 x = \dfrac{1 - \cos 2x}{2}$ $\quad\Big|\because \cos 2x = 1 - 2\sin^2 x$

and $\cos^2 x = \dfrac{1 + \cos 2x}{2}$. $\quad\Big|\because \cos 2x = 2\cos^2 x - 1$

28. $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$.

29. $\sin 3x = 3 \sin x - 4 \sin^3 x$.

30. $\cos 3x = 4 \cos^3 x - 3 \cos x$.

31. $\tan 3x = \dfrac{3 \tan x - \tan^3 x}{1 - 3\tan^2 x}$.

32. $\sin 18° = \dfrac{\sqrt{5}-1}{4}$.

33. $\cos 36° = \dfrac{\sqrt{5}+1}{4}$.

34. $\cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$.

35. $\sin 36° = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$.

Formulae from 36 to 42 are for general solutions (*i.e.* values of the variable from – ∞ to ∞ satisfying a given T-equation).

36. $\sin x = 0 \quad \Rightarrow x = n\pi$, where $n \in$ Z.

37. $\cos x = 0 \quad \Rightarrow \quad x = (2n + 1)\dfrac{\pi}{2}$, where $n \in$ Z.

38. $\tan x = 0 \quad \Rightarrow \quad x = n\pi$, where $n \in$ Z.

39. $\sin x = \sin y \quad \Rightarrow \quad x = n\pi + (-1)^n y$, where $n \in$ Z.

40. $\cos x = \cos y \quad \Rightarrow \quad x = 2n\pi \pm y$, where $n \in$ Z.

41. $\tan x = \tan y \quad \Rightarrow \quad x = n\pi + y$, where $n \in$ Z.

42. $\left.\begin{array}{l} \sin^2 x = \sin^2 y \\ \cos^2 x = \cos^2 y \\ \tan^2 x = \tan^2 y \end{array}\right] \Rightarrow \quad x = n\pi \pm y$, where $n \in$ Z.

43. To solve the classic equation $a \cos x \pm b \sin x = c$, divide both sides by $\sqrt{a^2+b^2}$.

In any triangle ABC,

44. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$. (Sine formula)

45. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

$\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$. (Cosine formulae)

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

46. $a = b \cos C + c \cos B$.

$b = c \cos A + a \cos C$. (Projection formulae)

$c = a \cos B + b \cos A$.

TEXTBOOK QUESTIONS SOLVED

EXERCISE 3.1 (*Page No.: 54–55*)

1. Find the radian measures corresponding to the following degree measures:

(*i*) 25° (*ii*) – 47°30′ (*iii*) 240° (*iv*) 520°.

Sol. Since 180° = π radians ∴ $1° = \dfrac{\pi}{180}$ radians

(*i*) $25° = 25 \times \dfrac{\pi}{180}$ radians $= \dfrac{5\pi}{36}$ radians

∴ Radian measure of 25° is $\dfrac{5\pi}{36}$.

(*ii*) $-47°30' = -47\dfrac{1}{2}^{\circ} = -\dfrac{95}{2} \times \dfrac{\pi}{180}$ radians

$\left(\because 30' = \left(\dfrac{30}{60}\right)^{\circ} = \dfrac{1}{2}^{\circ}\right)$

$= -\dfrac{19\pi}{72}$ radians

∴ Radian measure of – 47°30′ is $-\dfrac{19\pi}{72}$.

(*iii*) $240° = 240 \times \dfrac{\pi}{180}$ radians $= \dfrac{4\pi}{3}$ radians

∴ Radian measure of 240° is $\dfrac{4\pi}{3}$.

(*iv*) $520^\circ = 520 \times \dfrac{\pi}{180}$ radians $= \dfrac{26\pi}{9}$ radians

$\therefore$ Radian measure of 520° is $\dfrac{26\pi}{9}$.

2. Find the degree measures corresponding to the following radian measure: $\left(\text{Use } \pi = \dfrac{22}{7}\right)$

(*i*) $\dfrac{11}{16}$ (*ii*) -4 (*iii*) $\dfrac{5\pi}{3}$ (*iv*) $\dfrac{7\pi}{6}$.

Sol. Since π radians $= 180^\circ$ $\therefore$ 1 radian $= \dfrac{180^\circ}{\pi}$

(*i*) $\dfrac{11}{16}$ radians $= \dfrac{11}{16} \times \dfrac{180^\circ}{\pi}$

$$= \left(\frac{11 \times 45}{4} \times \frac{7}{22}\right)^\circ = \frac{315^\circ}{8}$$

$$\begin{array}{r} 39^\circ \\ 8\overline{)315} \\ 24 \\ \hline 75 \\ 72 \\ \hline 3^\circ \\ \times 60 \\ \hline 180' \\ 8\overline{)180}\,22' \\ 16 \\ \hline 20 \\ 16 \\ \hline 4 \times 60 \\ 8\overline{)240}\,30'' \\ 240 \\ \hline \times \end{array}$$

Ans. $39^\circ, 22', 30''$

(*ii*) -4 radians $= -4 \times \dfrac{180^\circ}{\pi} = \left(-720 \times \dfrac{7}{22}\right)^\circ$

$$= -\frac{2520^\circ}{11}$$

$$\begin{array}{r|l} 11 & \overline{2520}\ (229^\circ \\ & 22 \\ \hline & 32 \\ & 22 \\ \hline & 100 \\ & 99 \\ \hline & 10 \\ & \times 60 \\ \hline 11 & \overline{60'}\ (5' \\ & 55 \\ \hline & 5' \\ & \times 60 \\ \hline 11 & \overline{300}\ (27'' \\ & 22 \\ \hline & 80' \\ & 77 \\ \hline & 3 \end{array}$$

Ans. 229°, 5', 27" approximately.

(*iii*) $\dfrac{5\pi}{3}$ radians $= \dfrac{5\pi}{3} \times \dfrac{180^\circ}{\pi} = 300^\circ$

(*iv*) $\dfrac{7\pi}{6}$ radians$= \dfrac{7\pi}{6} \times \dfrac{180^\circ}{\pi} = 210^\circ$.

3. A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second?

Sol. Number of revolutions in one minute = 360

$\Rightarrow$ Number of revolutions in one second $= \dfrac{360}{60} = 6$

Since 1 revolution = 360° = 2π radians

$\therefore$ 6 revolutions $= 6 \times 2\pi = 12\pi$ radians

$\Rightarrow$ Number of radians turned in one second $= 12\pi$.

4. Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm. $\left(\text{Use } \pi = \dfrac{22}{7}\right)$

Sol. Here, l = length of arc = 22 cm

r = radius of circle = 100 cm

Let θ be the angle subtended at the centre, then

$$\theta = \frac{l}{r} \text{ radians}$$

$$= \frac{22}{100} \text{ radians} = \frac{11}{50} \times \frac{180^\circ}{\pi}$$

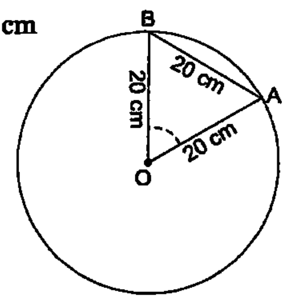
$$= \left(\frac{11 \times 18}{5} \times \frac{7}{22}\right)^\circ = \frac{63^\circ}{5}$$

$$\begin{array}{r} 12^\circ \\ 5\overline{)63} \\ \underline{60} \\ 3 \\ \times 60 \\ 5\overline{)180'}\,(36' \\ \underline{15} \\ 30 \\ \underline{30} \\ \times \end{array}$$

Ans. 12°, 36'

5. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc of the chord.

Sol. Radius of circle $= \frac{1}{2} \times 40$ cm $= 20$ cm



Chord AB = 20 cm

$\therefore$ In triangle OAB,

OA = OB = AB

$\Rightarrow$ Triangle is equilateral

$\Rightarrow$ Each angle = 60°

Let arc AB = l cm.

$$\text{Now, } \theta = \angle AOB = 60^\circ = 60 \times \frac{\pi}{180} = \frac{\pi}{3} \text{ radians}$$

$$r = 20 \text{ cm}$$

$$\therefore \quad l = r\theta = 20 \times \frac{\pi}{3} \text{ cm} \quad [l \text{ and } r \text{ have same units}]$$

$$= \frac{20\pi}{3} \text{ cm.}$$

6. If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii.

Sol. Let the radii of the two circles be r_1 and r_2 respectively. Also, let the length of arc in each case be l.

[∵ Arcs are of same length (given)]

For the first circle, $\theta = 60° = 60 \times \frac{\pi}{180} = \frac{\pi}{3}$ radians

We know that $l = r\,\theta$

∴ $r = \frac{l}{\theta}$

⇒ $r_1 = \frac{l}{\pi/3} = \frac{3l}{\pi}$ ...(*i*)

For the second circle, $\theta = 75° = 75 \times \frac{\pi}{180} = \frac{5\pi}{12}$ radians.

∴ $r_2 = \frac{l}{\theta} = \frac{l}{5\pi/12} = \frac{12l}{5\pi}$ ...(*ii*)

Dividing (*i*) by (*ii*), we get $\frac{r_1}{r_2} = \frac{3l}{\pi} \times \frac{5\pi}{12l} = \frac{5}{4}$

∴ $r_1 : r_2 = 5 : 4$.

7. Find the angle in radian through which a pendulum swings if its length is 75 cm and the tip describes an arc of length:

(*i*) 10 cm (*ii*) 15 cm (*iii*) 21 cm.

Sol. [If one end of an inelastic string is attached to a fixed point and to the other end is attached a heavy particle (called **bob**), then this system is called a pendulum. Length of the string is called the length of the pendulum.]

Here, $r = 75$ cm.

(*i*) $l = 10$ cm

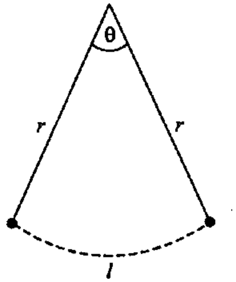
∴ $\theta = \frac{l}{r} = \frac{10}{75} = \frac{2}{15}$ radians.

(*ii*) $l = 15$ cm

∴ $\theta = \frac{l}{r} = \frac{15}{75} = \frac{1}{5}$ radians.

(*iii*) $l = 21$ cm

∴ $\theta = \frac{l}{r} = \frac{21}{75} = \frac{7}{25}$ radians.



EXERCISE 3.2 (*Page No.: 63*)

Find the values of other five trigonometric functions in Exercises 1 to 5.

1. $\cos x = -\frac{1}{2}$, **x lies in third quadrant.**

Sol. Given, $\cos x = -\frac{1}{2}$, x lies in third quadrant.

$$\therefore \quad \sec x = \frac{1}{\cos x} = -2 \quad ...(i)$$

Now, $\sin^2 x = 1 - \cos^2 x = 1 - \frac{1}{4} = \frac{3}{4}$

$$\Rightarrow \quad \sin x = \pm\frac{\sqrt{3}}{2}$$

Since x lies in third quadrant, $\sin x$ will be negative.

$$\therefore \quad \sin x = -\frac{\sqrt{3}}{2} \quad ...(ii)$$

and $\operatorname{cosec} x = \frac{1}{\sin x} = -\frac{2}{\sqrt{3}}$ ...(*iii*)

Also, $\tan x = \frac{\sin x}{\cos x} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$ ...(*iv*)

and $\cot x = \frac{1}{\tan x} = \frac{1}{\sqrt{3}}$ ...(*v*)

2. $\sin x = \frac{3}{5}$, **x lies in second quadrant.**

Sol. Given, $\sin x = \frac{3}{5}$, x lies in second quadrant.

$$\therefore \quad \operatorname{cosec} x = \frac{1}{\sin x} = \frac{5}{3} \quad ...(i)$$

Now, $\cos^2 x = 1 - \sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

$$\Rightarrow \quad \cos x = \pm \frac{4}{5}$$

Since x lies in second quadrant, $\cos x$ will be negative.

$$\therefore \quad \cos x = -\frac{4}{5} \qquad ...(ii)$$

$$\text{and} \quad \sec x = \frac{1}{\cos x} = -\frac{5}{4} \qquad ...(iii)$$

$$\text{Also,} \quad \tan x = \frac{\sin x}{\cos x} = \frac{\frac{3}{5}}{-\frac{4}{5}} = -\frac{3}{4} \qquad ...(iv)$$

$$\text{and } \cot x = \frac{1}{\tan x} = -\frac{4}{3}. \qquad ...(v)$$

3. $\cot x = \frac{3}{4}$, x lies in third quadrant.

Sol. Given, $\cot x = \frac{3}{4}$, x lies in third quadrant.

$$\therefore \quad \tan x = \frac{1}{\cot x} = \frac{4}{3} \qquad ...(i)$$

$$\text{Now, } \text{cosec}^2 x = 1 + \cot^2 x = 1 + \frac{9}{16} = \frac{25}{16}$$

$$\Rightarrow \quad \text{cosec } x = \pm \frac{5}{4}$$

Since x lies in third quadrant, cosec x will be negative.

$$\therefore \quad \text{cosec } x = -\frac{5}{4} \qquad ...(ii)$$

$$\text{and} \quad \sin x = \frac{1}{\text{cosec } x} = -\frac{4}{5} \qquad ...(iii)$$

$$\text{Also,} \quad \cos x = \frac{\cos x}{\sin x} \cdot \sin x = \cot x \sin x$$

$$= \frac{3}{4}\left(-\frac{4}{5}\right) = -\frac{3}{5} \qquad ...(iv)$$

$$\text{and} \quad \sec x = \frac{1}{\cos x} = -\frac{5}{3}. \qquad ...(v)$$

4. $\sec x = \frac{13}{5}$, x lies in fourth quadrant.

Sol. Given, $\sec x = \frac{13}{5}$, x lies in fourth quadrant.

$$\therefore \quad \cos x = \frac{1}{\sec x} = \frac{5}{13} \quad ...(i)$$

Now, $\sin^2 x = 1 - \cos^2 x = 1 - \frac{25}{169} = \frac{144}{169}$

$$\Rightarrow \quad \sin x = \pm \frac{12}{13}$$

Since x lies in fourth quadrant, $\sin x$ will be negative.

$$\therefore \quad \sin x = -\frac{12}{13} \quad ...(ii)$$

$$\text{and} \quad \operatorname{cosec} x = \frac{1}{\sin x} = -\frac{13}{12} \quad ...(iii)$$

$$\text{Also,} \quad \tan x = \frac{\sin x}{\cos x} = \frac{-\frac{12}{13}}{\frac{5}{13}} = -\frac{12}{5} \quad ...(iv)$$

$$\text{and} \quad \cot x = \frac{1}{\tan x} = -\frac{5}{12}. \quad ...(v)$$

5. $\tan x = -\frac{5}{12}$, x lies in second quadrant.

Sol. Given, $\tan x = -\frac{5}{12}$, x lies in second quadrant.

$$\therefore \quad \cot x = \frac{1}{\tan x} = -\frac{12}{5} \quad ...(i)$$

Now, $\sec^2 x = 1 + \tan^2 x = 1 + \frac{25}{144} = \frac{169}{144}$

$$\Rightarrow \quad \sec x = \pm \frac{13}{12}$$

Since x lies in second quadrant, $\sec x$ will be negative.

$$\therefore \quad \sec x = -\frac{13}{12} \quad ...(ii)$$

$$\text{and} \quad \cos x = \frac{1}{\sec x} = -\frac{12}{13} \quad ...(iii)$$

Also, $\sin x = \frac{\sin x}{\cos x} \cdot \cos x = \tan x \cos x$

$$= \left(-\frac{5}{12}\right)\left(-\frac{12}{13}\right) = \frac{5}{13} \qquad ...(iv)$$

and $\operatorname{cosec} x = \frac{1}{\sin x} = \frac{13}{5}$. ...(v)

Find the values of the trigonometric functions in Exercises 6 to 10.

6. sin 765°.

Sol. $\sin 765° = \sin (2 \times 360° + 45°) = \sin 45° = \frac{1}{\sqrt{2}}$

[$\because$ $\sin (n \times 360° + x) = \sin x, n \in Z$]

7. cosec (– 1410°).

Sol. $\operatorname{cosec} (-1410°) = \operatorname{cosec} (4 \times 360° - 1410°)$

$= \operatorname{cosec} (1440° - 1410°) = \operatorname{cosec} 30° = 2$

[$\because$ $\operatorname{cosec} x = \operatorname{cosec} (n \times 360° + x), n \in Z$]

8. $\tan \frac{19\pi}{3}$.

Sol. $\tan \frac{19\pi}{3} = \tan \left(\frac{18\pi + \pi}{3}\right) = \tan \left(6\pi + \frac{\pi}{3}\right) = \tan \frac{\pi}{3} = \sqrt{3}$

[$\because$ $\tan (n\pi + x) = \tan x, n \in Z$]

9. $\sin \left(-\frac{11\pi}{3}\right)$.

Sol. $\sin \left(-\frac{11\pi}{3}\right) = \sin \left(4\pi - \frac{11\pi}{3}\right)$

[$\because$ $\sin x = \sin (2n\pi + x), n \in Z$]

$= \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

10. $\cot \left(-\frac{15\pi}{4}\right)$.

Sol. $\cot \left(-\frac{15\pi}{4}\right) = \cot \left(4\pi - \frac{15\pi}{4}\right)$

[$\because$ $\cot x = \cot (n\pi + x), n \in Z$]

$= \cot \frac{\pi}{4} = 1$.

EXERCISE 3.3 (*Page No.: 73–74*)

1. Prove that: $\sin^2\frac{\pi}{6}+\cos^2\frac{\pi}{3}-\tan^2\frac{\pi}{4}=-\frac{1}{2}$.

Sol. We know that $\sin\frac{\pi}{6}=\frac{1}{2}$, $\cos\frac{\pi}{3}=\frac{1}{2}$, $\tan\frac{\pi}{4}=1$

$$\therefore \quad \text{L.H.S.} = \left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2-1^2=\frac{1}{4}+\frac{1}{4}-1$$

$$=\frac{1}{2}-1=-\frac{1}{2}=\text{R.H.S.}$$

2. Prove that: $2\sin^2\frac{\pi}{6}+\text{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3}=\frac{3}{2}$.

Sol. We know that $\sin\frac{\pi}{6}=\frac{1}{2}$

$$\text{cosec}\frac{7\pi}{6}=\text{cosec}\left(\frac{6\pi+\pi}{6}\right)=\text{cosec}\left(\pi+\frac{\pi}{6}\right)$$

$$=-\text{cosec}\frac{\pi}{6}=-2,\ \cos\frac{\pi}{3}=\frac{1}{2}$$

$$\therefore \quad \text{L.H.S.} = 2\left(\frac{1}{2}\right)^2+(-2)^2\left(\frac{1}{2}\right)^2$$

$$=\frac{1}{2}+1=\frac{3}{2}=\text{R.H.S.}$$

3. Prove that: $\cot^2\frac{\pi}{6}+\text{cosec}\frac{5\pi}{6}+3\tan^2\frac{\pi}{6}=6$.

Sol. We know that $\cot\frac{\pi}{6}=\sqrt{3}$

$$\text{cosec}\frac{5\pi}{6}=\text{cosec}\left(\frac{6\pi-\pi}{6}\right)=\text{cosec}\left(\pi-\frac{\pi}{6}\right)$$

$$=\text{cosec}\frac{\pi}{6}=2,\ \tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$$

$\therefore$ L.H.S. $= (\sqrt{3})^2 + 2 + 3\left(\frac{1}{\sqrt{3}}\right)^2 = 3 + 2 + 1$

$= 6 = $ R.H.S.

4. Prove that: $2\sin^2\frac{3\pi}{4} + 2\cos^2\frac{\pi}{4} + 2\sec^2\frac{\pi}{3} = 10$.

Sol. We know that $\sin\frac{3\pi}{4} = \sin\left(\frac{4\pi-\pi}{4}\right) = \sin\left(\pi - \frac{\pi}{4}\right)$

$$= \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \sec\frac{\pi}{3} = 2$$

$\therefore$ L.H.S. $= 2\left(\frac{1}{\sqrt{2}}\right)^2 + 2\left(\frac{1}{\sqrt{2}}\right)^2 + 2(2)^2$

$= 1 + 1 + 8 = 10 = $ R.H.S.

5. Find the value of:

(i) sin 75°

(ii) tan 15°.

Sol. (i) $\sin 75° = \sin(45° + 30°)$

$= \sin 45° \cos 30° + \cos 45° \sin 30°$

$[\because \sin(x + y) = \sin x \cos y + \cos x \sin y]$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

(ii) $\tan 15° = \tan(45° - 30°) = \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$

$$\left[\because \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}\right]$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \times \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

Rationalising

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{(\sqrt{3}-1)^2}{(\sqrt{3})^2 - 1^2}$$

$$= \frac{3+1-2\sqrt{3}}{3-1} = \frac{4-2\sqrt{3}}{2} = 2 - \sqrt{3}.$$

6. Prove that:

$$\cos\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-y\right) - \sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right) = \sin(x+y).$$

Sol. L.H.S. $= \cos\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-y\right) - \sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right)$

$=$ Put $\frac{\pi}{4}-x = A$ and $\frac{\pi}{4}-y = B$

$\therefore$ L.H.S. $= \cos A \cos B - \sin A \sin B = \cos(A+B)$

$$= \cos\left(\frac{\pi}{4} - x + \frac{\pi}{4} - y\right) = \cos\left(\frac{2\pi}{4} - (x+y)\right)$$

$$= \cos\left[\frac{\pi}{2} - (x+y)\right]$$

$= \sin(x+y) =$ R.H.S. $\quad \left[\because \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta\right]$

7. Prove that: $\dfrac{\tan\left(\frac{\pi}{4}+x\right)}{\tan\left(\frac{\pi}{4}-x\right)} = \left(\dfrac{1+\tan x}{1-\tan x}\right)^2.$

Sol. L.H.S. $= \dfrac{\tan\left(\frac{\pi}{4}+x\right)}{\tan\left(\frac{\pi}{4}-x\right)} = \dfrac{\dfrac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}.\tan x}}{\dfrac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}.\tan x}} = \dfrac{\dfrac{1+\tan x}{1-\tan x}}{\dfrac{1-\tan x}{1+\tan x}}$

$$= \frac{1+\tan x}{1-\tan x} \times \frac{1+\tan x}{1-\tan x} = \left(\frac{1+\tan x}{1-\tan x}\right)^2 = \text{R.H.S.}$$

8. Prove that: $\dfrac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)} = \cot^2 x.$

$$\text{III} \qquad \text{IV}$$

Sol. L.H.S. $= \dfrac{\cos(\pi + x)\cos(-x)}{\sin(\pi - x)\cos\left(\dfrac{\pi}{2} + x\right)} = \dfrac{(-\cos x)\cos x}{\sin x(-\sin x)}$

$$\text{II} \qquad \text{II}$$

$$= \frac{\cos^2 x}{\sin^2 x} = \cot^2 x = \text{R.H.S.}$$

9. Prove that:

$$\cos\left(\frac{3\pi}{2} + x\right)\cos(2\pi + x)\left[\cot\left(\frac{3\pi}{2} - x\right) + \cot(2\pi + x)\right] = 1.$$

Sol. L.H.S. $= \cos\left(\dfrac{3\pi}{2} + x\right)\cos(2\pi + x)$

$$\text{IV} \qquad \text{I}$$

$$\left[\cot\left(\frac{3\pi}{2} - x\right) + \cot(2\pi + x)\right]$$

$$\text{III} \qquad \text{I}$$

$$= \sin x \cos x [\tan x + \cot x]$$

$$= \sin x \cos x \left[\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right]$$

$$= \sin x \cos x \left[\frac{\sin^2 x + \cos^2 x}{\cos x \sin x}\right] = 1 = \text{R.H.S.}$$

10. Prove that:

$\sin(n+1)x \sin(n+2)x + \cos(n+1)x \cos(n+2)x = \cos x.$

Sol. L.H.S. $= \sin(n+1)x\sin(n+2)x + \cos(n+1)x\cos(n+2)x$

$= \text{Put } (n+1)x = A \text{ and } (n+2)x = B$

$\therefore$ L.H.S. $= \sin A \sin B + \cos A \cos B$

$= \cos A \cos B + \sin A \sin B = \cos(A - B)$

$= \cos[(n+1)x - (n+2)x] = \cos(nx + x - nx - 2x)$

$= \cos(-x) = \cos x = \text{R.H.S.}$

IV

11. Prove that:

$$\cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) = -\sqrt{2}\sin x.$$

Sol. L.H.S. $= \cos\left(\frac{3\pi}{4}+x\right) - \cos\left(\frac{3\pi}{4}-x\right)$

$= \cos\frac{3\pi}{4}\cos x - \sin\frac{3\pi}{4}\sin x - \left(\cos\frac{3\pi}{4}\cos x + \sin\frac{3\pi}{4}\sin x\right)$

[$\because$ cos (A + B) = cos A cos B – sin A sin B and
cos (A – B) = cos A cos B + sin A sin B]

$= \cos\frac{3\pi}{4}\cos x - \sin\frac{3\pi}{4}\sin x - \cos\frac{3\pi}{4}\cos x - \sin\frac{3\pi}{4}\sin x$

$= -2\sin\frac{3\pi}{4}\sin x = -2\sin\left(\frac{4\pi-\pi}{4}\right)\sin x$

$= -2\sin\left(\pi-\frac{\pi}{4}\right)\sin x = -2\sin\frac{\pi}{4}\sin x = -2.\frac{1}{\sqrt{2}}\sin x$

$= -\sqrt{2}\sin x$

12. Prove that: $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$.

Sol. L.H.S. $= \sin^2 6x - \sin^2 4x$

$= \sin(6x+4x)\sin(6x-4x)$

[$\because$ $\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$]

$= \sin 10x \sin 2x$ = R.H.S.

13. Prove that: $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$.

Sol. L.H.S. $= \cos^2 2x - \cos^2 6x$

[**Remember:** $\cos^2 A - \cos^2 B \neq \cos(A+B)\cos(A-B)$]

$= (1-\sin^2 2x) - (1-\sin^2 6x)$ [$\because$ $\cos^2\theta = 1-\sin^2\theta$]

$= \sin^2 6x - \sin^2 2x$

$= \sin(6x+2x)\sin(6x-2x)$

[$\because$ $\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$]

$= \sin 8x \sin 4x$ = R.H.S.

14. Prove that: $\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \sin 4x$.

Sol. L.H.S. $= (\sin 6x + \sin 2x) + 2\sin 4x$

$= 2\sin\frac{6x+2x}{2}\cos\frac{6x-2x}{2} + 2\sin 4x$

$\left[\because \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right]$

$= 2\sin 4x\cos 2x + 2\sin 4x$

Taking 2 sin $4x$ common,

$= 2 \sin 4x (\cos 2x + 1)$

$= 2 \sin 4x (2 \cos^2 x - 1 + 1)$

$= 4 \cos^2 x \sin 4x =$ R.H.S.

15. Prove that: $\cot 4x (\sin 5x + \sin 3x) = \cot x (\sin 5x - \sin 3x)$.

Sol. L.H.S. $= \cot 4x (\sin 5x + \sin 3x)$

$$= \cot 4x \,.\, 2 \sin \frac{5x+3x}{2} \cos \frac{5x-3x}{2}$$

$$\left[\because \quad \sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}\right]$$

$$= \frac{\cos 4x}{\sin 4x} \,.\, 2 \sin 4x \cos x$$

$= 2 \cos 4x \cos x.$ ...(*i*)

R.H.S. $= \cot x (\sin 5x - \sin 3x)$

$$= \cot x \,.\, 2 \cos \frac{5x+3x}{2} \sin \frac{5x-3x}{2}$$

$$\left[\because \quad \sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}\right]$$

$$= \frac{\cos x}{\sin x} \,.\, 2 \cos 4x \sin x$$

$= 2 \cos x \cos 4x.$ ...(*ii*)

From (*i*) and (*ii*), we get

L.H.S. = R.H.S. [$\because$ Each $= 2 \cos 4x \cos x$]

16. Prove that: $\dfrac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\dfrac{\sin 2x}{\cos 10x}$.

Sol. L.H.S. $= \dfrac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x}$

$$= \frac{-2 \sin \frac{9x+5x}{2} \sin \frac{9x-5x}{2}}{2 \cos \frac{17x+3x}{2} \sin \frac{17x-3x}{2}}$$ [Using C – D formula]

$$= -\frac{\sin 7x \sin 2x}{\cos 10x \sin 7x} = -\frac{\sin 2x}{\cos 10x} = \text{R.H.S.}$$

17. Prove that: $\dfrac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x.$

Sol. L.H.S. $= \dfrac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x}$

$$= \frac{2 \sin \frac{5x+3x}{2} \cos \frac{5x-3x}{2}}{2 \cos \frac{5x+3x}{2} \cos \frac{5x-3x}{2}} \quad \text{[Using C – D formulae]}$$

$$= \frac{\sin 4x \cos x}{\cos 4x \cos x} = \frac{\sin 4x}{\cos 4x} = \tan 4x = \text{R.H.S.}$$

18. Prove that: $\dfrac{\sin x - \sin y}{\cos x + \cos y} = \tan \dfrac{x-y}{2}.$

Sol. L.H.S. $= \dfrac{\sin x - \sin y}{\cos x + \cos y}$

$$= \frac{2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}}{2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}} \quad \text{[Using C – D formulae]}$$

Cancelling $2 \cos \dfrac{x+y}{2}$,

$$= \frac{\sin \frac{x-y}{2}}{\cos \frac{x-y}{2}} = \tan \frac{x-y}{2} = \text{R.H.S.}$$

19. Prove that: $\dfrac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x.$

Sol. L.H.S. $= \dfrac{\sin x + \sin 3x}{\cos x + \cos 3x}$

$$= \frac{2 \sin \frac{x+3x}{2} \cos \frac{x-3x}{2}}{2 \cos \frac{x+3x}{2} \cos \frac{x-3x}{2}} \quad \text{[Using C – D formulae]}$$

$$= \frac{\sin 2x \cos (-x)}{\cos 2x \cos (-x)} = \tan 2x = \text{R.H.S.}$$

20. Prove that: $\dfrac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2 \sin x.$

Sol. L.H.S. $= \dfrac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = \dfrac{-(\sin 3x - \sin x)}{-(\cos^2 x - \sin^2 x)}$

$$= \frac{\sin 3x - \sin x}{\cos^2 x - \sin^2 x}$$

$$= \frac{2\cos\frac{3x+x}{2}\sin\frac{3x-x}{2}}{\cos 2x} \quad \text{[Using C – D formulae]}$$

$[\because \cos^2 x - \sin^2 x = \cos 2x]$

$$= \frac{2\cos 2x \sin x}{\cos 2x} = 2 \sin x = \text{R.H.S.}$$

21. Prove that: $\dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x.$

Sol. L.H.S. $= \dfrac{(\cos 4x + \cos 2x) + \cos 3x}{(\sin 4x + \sin 2x) + \sin 3x}$ $\quad (\because 4x + 2x = 2 \times 3x)$

$$= \frac{2\cos\frac{4x+2x}{2}\cos\frac{4x-2x}{2} + \cos 3x}{2\sin\frac{4x+2x}{2}\cos\frac{4x-2x}{2} + \sin 3x}$$

[Using C – D formulae]

$$= \frac{2\cos 3x \cos x + \cos 3x}{2\sin 3x \cos x + \sin 3x}$$

Taking cos $3x$ common both from numerator and denominator.

$$= \frac{\cos 3x\,(2\cos x + 1)}{\sin 3x\,(2\cos x + 1)} = \cot 3x = \text{R.H.S.}$$

22. Prove that: $\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$.

Sol. We know that $3x = 2x + x$

$\therefore \quad \cot 3x = \cot (2x + x)$

$$\Rightarrow \quad \cot 3x = \frac{\cot 2x \cot x - 1}{\cot x + \cot 2x}$$

$$\left[\because \quad \cot (A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}\right]$$

cross-multiplying

$$\cot 3x (\cot x + \cot 2x) = \cot 2x \cot x - 1$$

$\Rightarrow$ $\cot 3x \cot x + \cot 3x \cot 2x = \cot 2x \cot x - 1$

$\Rightarrow$ $\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1.$

23. Prove that: $\tan 4x = \dfrac{4 \tan x (1 - \tan^2 x)}{1 - 6 \tan^2 x + \tan^4 x}$.

Sol. Since $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

Put $A = 2x$,

$$\therefore \quad \tan 4x = \frac{2 \tan 2x}{1 - \tan^2 2x} = \frac{2\left(\dfrac{2 \tan x}{1 - \tan^2 x}\right)}{1 - \left(\dfrac{2 \tan x}{1 - \tan^2 x}\right)^2}$$

$$= \frac{\dfrac{4 \tan x}{1 - \tan^2 x}}{1 - \dfrac{4 \tan^2 x}{(1 - \tan^2 x)^2}}$$

$$= \frac{\dfrac{4 \tan x}{1 - \tan^2 x}}{\dfrac{(1 - \tan^2 x)^2 - 4 \tan^2 x}{(1 - \tan^2 x)^2}}$$

$$= \frac{4 \tan x (1 - \tan^2 x)}{1 - 2 \tan^2 x + \tan^4 x - 4 \tan^2 x}$$

$$= \frac{4 \tan x (1 - \tan^2 x)}{1 - 6 \tan^2 x + \tan^4 x}.$$

24. Prove that: $\cos 4x = 1 - 8 \sin^2 x \cos^2 x$.

Sol. Since $\cos 2A = 1 - 2 \sin^2 A$

Put $A = 2x$,

$$\therefore \quad \cos 4x = 1 - 2 \sin^2 2x$$

$$= 1 - 2(2 \sin x \cos x)^2$$

$$= 1 - 8 \sin^2 x \cos^2 x.$$

25. Prove that: $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$.

Sol. Since $\cos 2A = 2\cos^2 A - 1$

Put $A = 3x$,

$\therefore$ $\cos 6x = 2\cos^2 3x - 1$

$= 2(4\cos^3 x - 3\cos x)^2 - 1$

$= 2(16\cos^6 x - 24\cos^4 x + 9\cos^2 x) - 1$

$= 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1.$

EXERCISE 3.4 *(Page No.: 78)*

Note : For general solutions, it is customary to use the following relations:

$-\sin\alpha = \sin(\pi + \alpha)$, $-\cos\alpha = \cos(\pi - \alpha)$ and
$-\tan\alpha = \tan(\pi - \alpha)$

Find the principal and general solutions of the following equations:

1. $\tan x = \sqrt{3}$ **2. $\sec x = 2$**

3. $\cot x = -\sqrt{3}$ **4. $\operatorname{cosec} x = -2$**

Sol. 1. $\tan x = \sqrt{3}$

(*i*) Since $\tan x$ is positive, principal solutions (***i.e.* values of x between 0° and 360° = 2π**) are in first and third quadrants.

Now, $\tan x = \sqrt{3}$ III

$$= \tan\frac{\pi}{3} = \tan\left(\pi + \frac{\pi}{3}\right), \text{ i.e., } \tan\frac{4\pi}{3}$$

$\therefore$ Principal solutions are $\frac{\pi}{3}, \frac{4\pi}{3}$.

(*ii*) Since $\tan x = \sqrt{3} = \tan\frac{\pi}{3}$,

$\therefore$ $x = n\pi + \frac{\pi}{3}$,

$n \in Z$ is the general solution.

2. $\sec x = 2$ $\Rightarrow$ $\cos x = \frac{1}{2}$

(*i*) Since $\cos x$ is positive, principal solutions are in first and fourth quadrants.

Now, $\cos x = \frac{1}{2}$

$$= \cos \frac{\pi}{3} = \cos \overset{\text{IV}}{\left(2\pi - \frac{\pi}{3}\right)}, \text{ i.e., } \cos \frac{5\pi}{3}$$

$\therefore$ Principal solutions are $\frac{\pi}{3}, \frac{5\pi}{3}$.

(ii) Since $\cos x = \frac{1}{2} = \cos \frac{\pi}{3}$,

$$\therefore \quad x = 2n\pi \pm \frac{\pi}{3},$$

$n \in \mathbf{Z}$ is the general solution.

3. $\cot x = -\sqrt{3} \Rightarrow \tan x = -\frac{1}{\sqrt{3}}$

(i) Since tan x is negative, principal solutions are in second and fourth quadrants.

Now, $\tan x = -\frac{1}{\sqrt{3}} = -\tan \frac{\pi}{6}$

$$= \tan \overset{\text{II}}{\left(\pi - \frac{\pi}{6}\right)}, \text{ i.e., } \tan \frac{5\pi}{6}$$

and $$\tan \overset{\text{IV}}{\left(2\pi - \frac{\pi}{6}\right)}, \text{ i.e., } \tan \frac{11\pi}{6}$$

$\therefore$ Principal solutions are $\frac{5\pi}{6}, \frac{11\pi}{6}$.

(ii) Since $\tan x = -\frac{1}{\sqrt{3}} = -\tan \frac{\pi}{6}$

$$= \tan \left(\pi - \frac{\pi}{6}\right) = \tan \frac{5\pi}{6}$$

$$\therefore \quad x = n\pi + \frac{5\pi}{6},$$

$n \in \mathbf{Z}$ is the general solution.

4. cosec $x = -2 \quad \Rightarrow \quad \sin x = -\dfrac{1}{2}$

(*i*) Since sin x is negative, principal solutions are in third and fourth quadrants.

Now, $\sin x = -\dfrac{1}{2} = -\sin\dfrac{\pi}{6}$

$$\overset{\text{III}}{= \sin\left(\pi + \frac{\pi}{6}\right)}, \text{ i.e., } \sin\frac{7\pi}{6}$$

and $$= \sin\underset{\text{IV}}{\left(2\pi - \frac{\pi}{6}\right)}, \text{ i.e., } \sin\frac{11\pi}{6}.$$

$\therefore$ Principal solutions are $\dfrac{7\pi}{6}, \dfrac{11\pi}{6}$.

(*ii*) Since $\sin x = -\dfrac{1}{2} = -\sin\dfrac{\pi}{6} = \sin\left(\pi + \dfrac{\pi}{6}\right)$

$$= \sin\frac{7\pi}{6}$$

$\therefore \quad x = n\pi + (-1)^n \cdot \dfrac{7\pi}{6}$,

$n \in Z$ is the general solution.

Find the general solution for each of the following equations:

5. $\cos 4x = \cos 2x$ **6.** $\cos 3x + \cos x - \cos 2x = 0$

7. $\sin 2x + \cos x = 0$ **8.** $\sec^2 2x = 1 - \tan 2x$

9. $\sin x + \sin 3x + \sin 5x = 0$.

Sol.

5. Given equation is $\cos 4x = \cos 2x$

or $\cos 4x - \cos 2x = 0$

or $-2 \sin\dfrac{4x + 2x}{2} \sin\dfrac{4x - 2x}{2} = 0$

$$\left[\because \cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

or $\sin 3x \sin x = 0$

$\Rightarrow \quad \sin 3x = 0 \quad$ or $\quad \sin x = 0$

$\Rightarrow \quad 3x = n\pi \quad$ or $\quad x = n\pi, \quad n \in Z$

$\therefore \quad x = \dfrac{n\pi}{3}, n\pi, \quad n \in Z.$

6. Given equation is $\cos 3x + \cos x - \cos 2x = 0$

or $2\cos\frac{3x+x}{2}\cos\frac{3x-x}{2} - \cos 2x = 0$ $(3x + x = 2 \times 2x)$

$\left[\because \cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}\right]$

or $2\cos 2x \cos x - \cos 2x = 0$

Taking $\cos 2x$ common, $\cos 2x(2\cos x - 1) = 0$

$\Rightarrow$ either $\cos 2x = 0$ or $2\cos x - 1 = 0$

$\Rightarrow$ $2x = (2n+1)\frac{\pi}{2}, n \in Z$ or $\cos x = \frac{1}{2}$

$\Rightarrow$ $x = (2n+1)\frac{\pi}{4}, n \in Z$ or $\cos x = \cos\frac{\pi}{3}$

$\therefore$ $x = (2n+1)\frac{\pi}{4}$ or $x = 2n\pi \pm \frac{\pi}{3}, n \in Z$.

7. Given equation is $\sin 2x + \cos x = 0$

or $2\sin x\cos x + \cos x = 0$

or $\cos x(2\sin x + 1) = 0$

or either $\cos x = 0$ or $2\sin x + 1 = 0$

$\Rightarrow$ $x = (2n+1)\frac{\pi}{2}, n \in Z$ or $\sin x = -\frac{1}{2} = -\sin\frac{\pi}{6}$

$= \sin\left(\pi + \frac{\pi}{6}\right) = \sin\frac{7\pi}{6}$

$\Rightarrow$ $x = (2n+1)\frac{\pi}{2}$ or $x = n\pi + (-1)^n \cdot \frac{7\pi}{6}, n \in Z$.

8. Given equation is $\sec^2 2x = 1 - \tan 2x$

or $1 + \tan^2 2x = 1 - \tan 2x$

$|\because \sec^2\theta = 1 + \tan^2\theta$

or $\tan^2 2x + \tan 2x = 0$

or $\tan 2x(\tan 2x + 1) = 0$

$\Rightarrow$ either $\tan 2x = 0$ or $\tan 2x + 1 = 0$

$\Rightarrow$ $2x = n\pi, n \in Z$ or $\tan 2x = -1 = -\tan\frac{\pi}{4}$

$= \tan\left(\pi - \frac{\pi}{4}\right) = \tan\frac{3\pi}{4}$

$$\Rightarrow \qquad x = \frac{n\pi}{2}, \qquad \text{or} \qquad 2x = n\pi + \frac{3\pi}{4}, n \in Z$$

$$\Rightarrow \qquad x = \frac{n\pi}{2}, \qquad \text{or} \qquad x = \frac{n\pi}{2} + \frac{3\pi}{8}, n \in Z.$$

9. Given equation is $\sin x + \sin 3x + \sin 5x = 0$

or $(\sin 5x + \sin x) + \sin 3x = 0$

$(\because 5x + x = 2 \times 3x)$

Applying $(\sin C + \sin D) = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

or $$2 \sin \frac{5x+x}{2} \cos \frac{5x-x}{2} + \sin 3x = 0$$

or $2 \sin 3x \cos 2x + \sin 3x = 0$

Taking $\sin 3x$ common

$\sin 3x\,(2 \cos 2x + 1) = 0$

$\Rightarrow$ either $\sin 3x = 0$ or $2 \cos 2x + 1 = 0$

$$\Rightarrow 3x = n\pi, n \in Z \qquad \text{or} \qquad \cos 2x = -\frac{1}{2} = -\cos\frac{\pi}{3}$$

$$= \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$$

$$\Rightarrow x = \frac{n\pi}{3}, n \in Z \qquad \text{or} \qquad 2x = 2n\pi \pm \frac{2\pi}{3}, n \in Z$$

$$\Rightarrow x = \frac{n\pi}{3} \qquad \text{or} \qquad x = n\pi \pm \frac{\pi}{3}, n \in Z.$$

MISCELLANEOUS EXERCISE ON CHAPTER 3

(Page No.: 81–82)

1. Prove that: $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0.$

Sol. $$\text{L.H.S.} = \cos\left(\frac{\pi}{13} + \frac{9\pi}{13}\right) + \cos\left(\frac{\pi}{13} - \frac{9\pi}{13}\right) + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$$

$[\because 2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)]$

$$= \cos \frac{10\pi}{13} + \cos \left(\frac{-8\pi}{13}\right) + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13}$$

$$= \cos \frac{10\pi}{13} + \cos \frac{8\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13}$$

[$\because \cos(-\theta) = \cos\theta$]

$$= \cos \frac{13\pi - 3\pi}{13} + \cos \frac{13\pi - 5\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13}$$

$$= \cos \left(\pi - \frac{3\pi}{13}\right) + \cos \left(\pi - \frac{5\pi}{13}\right) + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13}$$

II II

$$= -\cos \frac{3\pi}{13} - \cos \frac{5\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13}$$

[$\because \cos(\pi - \theta) = -\cos\theta$

$= 0 =$ R.H.S.

2. Prove that:

$(\sin 3x + \sin x) \sin x + (\cos 3x - \cos x) \cos x = 0.$

Sol. L.H.S. $= (\sin 3x + \sin x) \sin x + (\cos 3x - \cos x) \cos x$

$$= 2 \sin \frac{3x + x}{2} \cos \frac{3x - x}{2} \sin x$$

$$- 2 \sin \frac{3x + x}{2} \sin \frac{3x - x}{2} \cos x$$

[Using C – D formulae]

$= 2 \sin 2x \cos x \sin x - 2 \sin 2x \sin x \cos x$

$= 0 =$ R.H.S.

3. Prove that:

$(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4 \cos^2 \frac{x+y}{2}.$

Sol. L.H.S. $= (\cos x + \cos y)^2 + (\sin x - \sin y)^2$

$$= \left(2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}\right)^2 + \left(2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}\right)^2$$

[Using C – D formulae]

$$= 4 \cos^2 \frac{x+y}{2} \cos^2 \frac{x-y}{2} + 4 \cos^2 \frac{x+y}{2} \sin^2 \frac{x-y}{2}$$

Taking $4\cos^2\frac{x+y}{2}$ common,

$$= 4\cos^2\frac{x+y}{2}\left(\cos^2\frac{x-y}{2}+\sin^2\frac{x-y}{2}\right)$$

(using $\cos^2\theta + \sin^2\theta = 1$)

$$= 4\cos^2\frac{x+y}{2} \times 1 = 4\cos^2\frac{x+y}{2} = \text{R.H.S.}$$

4. Prove that:

$$(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4\sin^2\frac{x-y}{2}.$$

Sol. L.H.S. = $(\cos x - \cos y)^2 + (\sin x - \sin y)^2$

$$= \left(-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}\right)^2 + \left(2\cos\frac{x+y}{2}\sin\frac{x-y}{2}\right)^2$$

[Using C – D formulae]

$$= 4\sin^2\frac{x+y}{2}\sin^2\frac{x-y}{2} + 4\cos^2\frac{x+y}{2}\sin^2\frac{x-y}{2}$$

$$= 4\sin^2\frac{x-y}{2}\left(\sin^2\frac{x+y}{2}+\cos^2\frac{x+y}{2}\right)$$

$$= 4\sin^2\frac{x-y}{2} \times 1 \qquad (\because \sin^2\theta + \cos^2\theta = 1)$$

$$= 4\sin^2\frac{x-y}{2} = \text{R.H.S.}$$

5. Prove that:

$$\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x \cos 2x \sin 4x.$$

Sol. L.H.S. = $\sin x + \sin 3x + \sin 5x + \sin 7x$

$= (\sin 7x + \sin x) + (\sin 5x + \sin 3x)$ | $\because 7x + x = 5x + 3x$

$$= 2\sin\frac{7x+x}{2}\cos\frac{7x-x}{2} + 2\sin\frac{5x+3x}{2}\cos\frac{5x-3x}{2}$$

[Using C – D formulae]

$= 2\sin 4x \cos 3x + 2\sin 4x \cos x$

Taking $2\sin 4x$ common.

$= 2\sin 4x(\cos 3x + \cos x)$

$$= 2 \sin 4x \,.\, 2 \cos \frac{3x + x}{2} \cos \frac{3x - x}{2}$$

[Using C – D formulae]

$= 4 \sin 4x \cos 2x \cos x$

$= 4 \cos x \cos 2x \sin 4x = \text{R.H.S.}$

6. Prove that: $\dfrac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x.$

Sol. L.H.S. $= \dfrac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)}$

$$= \frac{2 \sin \frac{7x + 5x}{2} \cos \frac{7x - 5x}{2} + 2 \sin \frac{9x + 3x}{2} \cos \frac{9x - 3x}{2}}{2 \cos \frac{7x + 5x}{2} \cos \frac{7x - 5x}{2} + 2 \cos \frac{9x + 3x}{2} \cos \frac{9x - 3x}{2}}$$

[Using C – D formulae]

$$= \frac{2 \sin 6x \cos x + 2 \sin 6x \cos 3x}{2 \cos 6x \cos x + 2 \cos 6x \cos 3x}$$

Taking 2 sin 6x common from numerator and 2 cos 6x common from denominator,

$$= \frac{2 \sin 6x (\cos x + \cos 3x)}{2 \cos 6x (\cos x + \cos 3x)}$$

$$= \frac{\sin 6x}{\cos 6x} = \tan 6x = \text{R.H.S.}$$

7. Prove that:

$$\sin 3x + \sin 2x - \sin x = 4 \sin x \cos \frac{x}{2} \cos \frac{3x}{2}.$$

Sol. L.H.S. $= \sin 3x + \sin 2x - \sin x$

$= (\sin 3x - \sin x) + \sin 2x$

$= 2 \cos \dfrac{3x + x}{2} \sin \dfrac{3x - x}{2} + \sin 2x$ [Using C – D formulae]

$= 2 \cos 2x \sin x + 2 \sin x \cos x$

$= 2 \sin x (\cos 2x + \cos x)$

$$= 2 \sin x \,.\, 2 \cos \frac{2x + x}{2} \cos \frac{2x - x}{2}$$

[Using C – D formulae]

$$= 4 \sin x \cos \frac{x}{2} \cos \frac{3x}{2} = \text{R.H.S.}$$

Find $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ in each of the following in Exercises 8 to 10:

8. $\tan x = -\frac{4}{3}$, x in quadrant II.

Sol. $\tan x = -\frac{4}{3}$, (x is in qudrant II *i.e.* $\frac{\pi}{2} < x < \pi$)

$$\Rightarrow \quad \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$$

i.e., $\frac{x}{2}$ lies in the first quadrant, so that all t-ratios of $\frac{x}{2}$ are positive.

Now $\sec^2 x = 1 + \tan^2 x = 1 + \frac{16}{9} = \frac{9+16}{9} = \frac{25}{9}$

$\therefore \sec x = \pm\frac{5}{3}$

But x is in IInd Quadrant

$\therefore \sec x$ is negative and $= -\frac{5}{3} \Rightarrow \cos x = -\frac{3}{5}$

We know that $\sin^2 \frac{x}{2} = \frac{1-\cos x}{2}$ $\quad \left| \because \sin^2\theta = \frac{1-\cos 2\theta}{2} \right.$

$$\therefore \quad \sin \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}} = \pm\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}}$$

$$= \pm\sqrt{\frac{4}{5}} = \pm\frac{2}{\sqrt{5}}$$

But $\frac{x}{2}$ is in the first quadrant, therefore $\sin \frac{x}{2}$ is positive

and $= \frac{2}{\sqrt{5}}$

We know that $\cos^2 \frac{x}{2} = \frac{1+\cos x}{2}$ $\quad \left| \because \cos^2\theta = \frac{1+\cos 2\theta}{2} \right.$

$$\therefore \quad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}} = \pm\sqrt{\frac{1-\frac{3}{5}}{2}} = \pm\frac{1}{\sqrt{5}}$$

$$\therefore \quad \cos\frac{x}{2} = \frac{1}{\sqrt{5}} \quad \left(\because \frac{x}{2} \text{ is in Ist quadrant}\right)$$

$$\therefore \quad \tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\frac{2}{\sqrt{5}}}{\frac{1}{\sqrt{5}}} = 2.$$

9. $\cos x = -\frac{1}{3}$, x in quadrant III.

Sol. $\cos x = -\frac{1}{3}$, (x in quadrant III, *i.e.*, $\pi < x < \frac{3\pi}{2}$) (given)

$$\Rightarrow \quad \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4} \quad \Rightarrow \quad 90° < \frac{x}{2} < 135°$$

i.e., $\frac{x}{2}$ lies in the second quadrant, so that $\sin\frac{x}{2} > 0$, $\cos\frac{x}{2} < 0$ and $\tan\frac{x}{2} < 0$.

We know that

$$\sin^2\frac{x}{2} = \frac{1-\cos x}{2}$$

$$\therefore \quad \sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}} = \pm\sqrt{\frac{1-\left(-\frac{1}{3}\right)}{2}} = \pm\sqrt{\frac{2}{3}}$$

$$\therefore \quad \sin\frac{x}{2} = \sqrt{\frac{2}{3}}. \quad \left[\because \sin\frac{x}{2} > 0\right]$$

Again, $$\cos^2\frac{x}{2} = \frac{1+\cos x}{2}$$

$$\therefore \quad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\therefore \quad \cos\frac{x}{2} = -\sqrt{\frac{1+\cos x}{2}} \quad \left(\because \cos\frac{x}{2} < 0\right)$$

$$= -\sqrt{\frac{1-\frac{1}{3}}{2}} = -\sqrt{\frac{\frac{2}{3}}{2}} = -\sqrt{\frac{1}{3}} = -\frac{1}{\sqrt{3}}\,.$$

$$\therefore \qquad \tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\sqrt{\frac{2}{3}}}{-\frac{1}{\sqrt{3}}} = -\sqrt{2}\,.$$

10. $\sin x = \frac{1}{4}$, x in quadrant II.

Sol. $\sin x = \frac{1}{4}$, (x in quadrant II, *i.e.*, $\frac{\pi}{2} < x < \pi$)

$$\Rightarrow \qquad \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$$

i.e., $\frac{x}{2}$ lies in the first quadrant, so that all t- ratios of $\frac{x}{2}$ are positive.

Also, $\qquad \cos^2 x = 1 - \sin^2 x = 1 - \frac{1}{16} = \frac{15}{16}$

$$\therefore \qquad \cos x = \pm\frac{\sqrt{15}}{4}$$

and $\cos x$ is negative in the second quadrant. (x is given to be in the second quadrant)

$$\Rightarrow \qquad \cos x = -\frac{\sqrt{15}}{4}$$

We know that $\sin^2\frac{x}{2} = \frac{1-\cos x}{2}$

$$\therefore \qquad \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1+\frac{\sqrt{15}}{4}}{2}}$$

$$= \sqrt{\frac{4+\sqrt{15}}{8}}$$

$$= \sqrt{\frac{8+2\sqrt{15}}{16}} = \sqrt{\frac{5+3+2\sqrt{5}\sqrt{3}}{16}}$$

$$= \sqrt{\frac{\left(\sqrt{5}\right)^2 + \left(\sqrt{3}\right)^2 + 2\sqrt{5}\sqrt{3}}{16}} = \sqrt{\left(\frac{\sqrt{5}+\sqrt{3}}{4}\right)^2} = \frac{\sqrt{5}+\sqrt{3}}{4}.$$

Similarly, $\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}} = \sqrt{\frac{1-\frac{\sqrt{15}}{4}}{2}}$

$$= \sqrt{\frac{4-\sqrt{15}}{8}} = \frac{\sqrt{5}-\sqrt{3}}{4}$$

(For simplification, see the simplification for $\sin\frac{x}{2}$)

$\therefore$ $\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$

$$= \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} \times \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

(Rationalising)

$$= \frac{(\sqrt{5}+\sqrt{3})^2}{5-3} = \frac{5+3+2\sqrt{5}\sqrt{3}}{2} = \frac{8+2\sqrt{15}}{2} = 4+\sqrt{15}.$$

□□