4 Principle of Mathematical Induction

Lesson at a Glance

1. Deduction: It is the application of a general case (or result) to the particular case (or situation).

2. Induction: It means generalisation from particular cases or facts.

3. Principle of Mathematical Induction

To establish the truth of statements formulated in terms of n, where n is a positive integer, we use **the principle of mathematical induction** stated below: Suppose there is a given statement $P(n)$ involving the natural numbers n such that

(*i*) the statement is true for $n = 1$ *i.e.*, **P (1) is true** and

(*ii*) the truth of $P(k)$, where k is some positive integer, implies the truth of $P(k + 1)$ *i.e.*, **if the statement is true for $n = k$ (where k is some positive integer), then the statement is also true for $n = k + 1$.**

Then, $P(n)$ is true for all natural numbers n.

TEXTBOOK QUESTIONS SOLVED

EXERCISE 4.1 (*Page No.: 94–95*)

Prove the following by using the principle of mathematical induction for all $n \in$ N.

1. $1 + 3 + 3^2 + ... + 3^{n-1} = \dfrac{3^n - 1}{2}$.

Sol. Let $P(n) : 1 + 3 + 3^2 + ... + 3^{n-1} = \dfrac{3^n - 1}{2}$. ...(*i*)

Step I: Putting n = 1 in (*i*)

$P(1) : 1 = \dfrac{3^1 - 1}{2}$ or $1 = 1$ which is true.

Thus P(1) is true.

Step II: Assume that P(k) is true for some natural number k

i.e., let $1 + 3 + 3^2 + ... + 3^{k-1} = \frac{3^k - 1}{2}$ ...(*ii*)

Step III: We want to prove that [changing k to $k + 1$ in (*ii*)]

$P(k+1) : 1 + 3 + 3^2 + ... + 3^{k-1} + 3^k = \frac{3^{k+1} - 1}{2}$ is also true.

Now, L.H.S. of P(k + 1)

$$= 1 + 3 + 3^2 + ... + 3^{k-1} + 3^k = \frac{3^k - 1}{2} + 3^k \quad \text{[Using (ii)]}$$

$$= \frac{3^k - 1 + 2.3^k}{2} = \frac{(1+2)\,3^k - 1}{2} = \frac{3.3^k - 1}{2}$$

$$= \frac{3^{k+1} - 1}{2} = \text{R.H.S. of P } (k + 1)$$

$\Rightarrow$ P(k + 1) is true when P(k) is true.

Hence by P.M.I., P(n) is true for every positive integer n.

2. $1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Sol. Let $P(n) : 1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$ ...(*i*)

Step I: Putting n = 1 in (*i*), $P(1) : 1^3 = \left[\frac{1(1+1)}{2}\right]^2$ or

1 = 1 which is true.

Thus, P(1) is true.

Step II: Assume that P(k) is true for some natural number k.

i.e., let $1^3 + 2^3 + 3^3 + ... + k^3 = \left[\frac{k(k+1)}{2}\right]^2$ ...(*ii*)

Step III: We want to prove that [changing k to k+1 in (*ii*)]

$$P(k+1) : 1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3 = \left[\frac{(k+1)(k+2)}{2}\right]^2$$

is also true

Now, L.H.S. of P(k + 1) = $1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3$

$$= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 \qquad \text{[Using (ii)]}$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3 = (k+1)^2\left[\frac{k^2}{4} + (k+1)\right]$$

$$= (k+1)^2\left[\frac{k^2+4k+4}{4}\right] = (k+1)^2\,\frac{(k+2)^2}{4}$$

[$\because$ $k^2 + 4k + 4 = k^2 + 2k + 2k + 4 = k(k+2) + 2(k+2)$

$= (k+2)(k+2) = (k+2)^2$)]

$$= \left[\frac{(k+1)(k+2)}{2}\right]^2 = \text{R.H.S. of } P(k+1)$$

$\Rightarrow$ P(k + 1) is true when P(k) is true.

Hence by P.M.I., P(n) is true for every positive integer n.

3. $1 + \dfrac{1}{1+2} + \dfrac{1}{1+2+3} + \ldots + \dfrac{1}{1+2+3+\ldots+n} = \dfrac{2n}{n+1}$.

Sol. Let P(n) : $1 + \dfrac{1}{1+2} + \dfrac{1}{1+2+3} + \ldots + \dfrac{1}{1+2+3+\ldots+n}$

$$= \frac{2n}{n+1}$$

Step I: Putting n = 1 in (i), ...(i)

P(1) : $1 = \dfrac{2\times1}{1+1}$ or $1 = 1$ which is true.

Thus, P(1) is true

Step II: Assume that P(k) is true for some natural number k.

i.e., let $1 + \dfrac{1}{1+2} + \dfrac{1}{1+2+3} + \ldots + \dfrac{1}{1+2+3+\ldots+k}$

$$= \frac{2k}{k+1} \qquad \ldots(ii)$$

Step III: We want to prove that [changing k to k+1 in (*ii*)]

$$P(k+1) : 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+k} + \frac{1}{1+2+3+\ldots+k+(k+1)} = \frac{2(k+1)}{k+2}$$

is also true.

Now, L.H.S. of P (k+1) = $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+k} + \frac{1}{1+2+3+\ldots+k+(k+1)}$

$$= \frac{2k}{k+1} + \frac{1}{1+2+3+\ldots+k+(k+1)} \qquad \text{[Using (ii)]}$$

$$= \frac{2k}{k+1} + \frac{1}{\frac{k+1}{2}(1+k+1)}$$

$$\left[\because 1+2+3+\ldots+k+(k+1) \text{ is an A.P. of } (k+1) \text{ terms} \quad \therefore S_n = \frac{n}{2}(a+l) \Rightarrow S_{k+1} = \frac{k+1}{2}(1+k+1)\right]$$

$$= \frac{2k}{k+1} + \frac{2}{(k+1)(k+2)} = \frac{2}{k+1}\left(k + \frac{1}{k+2}\right)$$

$$= \frac{2}{k+1}\left(\frac{k^2+2k+1}{k+2}\right) = \frac{2}{k+1} \cdot \frac{(k+1)^2}{k+2}$$

$$= \frac{2(k+1)}{k+2} = \text{R.H.S. of } P(k+1)$$

$\Rightarrow$ P(k + 1) is true when P(k) is true.

Hence by P.M.I., P(n) is true for every positive integer n.

4. $1.2.3 + 2.3.4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$.

Sol. Let P(n) be the statement

"$1.2.3 + 2.3.4 + \ldots + n(n+1)(n+2)$

$= \frac{n(n+1)(n+2)(n+3)}{4}$" ...(*i*)

Step I: Then P (1) is the statement, [Putting $n = 1$ in (*i*)]

"$1.2.3 = \frac{1\times2\times3\times4}{4}$" which is true.

$\therefore$ P(1) is true

Step II: Suppose P (k) is true, [Putting $n = k$ in (*i*)]

i.e., let $1.2.3 + 2.3.4 + ... + k(k + 1)(k + 2)$

$$= \frac{k(k+1)(k+2)(k+3)}{4} \qquad ...(ii)$$

Step III: P (k + 1) is the statement, [Putting $n = k + 1$ in (*i*)]

"$1.2.3 + 2.3.4 + ... + k(k + 1)(k + 2) + (k + 1)(k + 2)(k + 3)$

$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$". We have to prove it.

Putting the value of L.H.S. of eqn. (*ii*) in L.H.S. of P(k + 1), we have

$1.2.3 + 2.3.4 + ... + k(k + 1)(k + 2) + (k + 1)(k + 2)(k + 3)$

$$= \frac{k(k+1)(k+2)(k+3)}{4} + (k + 1)(k + 2)(k + 3)$$

$$= \frac{(k+1)(k+2)(k+3)}{4}(k + 4) = \text{R.H.S. of } P(k + 1)$$

$\Rightarrow$ P (k + 1) is true.

$\therefore$ By the principle of mathematical induction, P (n) is true for all natural numbers n.

5. $\mathbf{1.3 + 2.3^2 + 3.3^3 + ... + n.3^n = \frac{(2n-1)3^{n+1}+3}{4}}$.

Sol. Let $P(n) : 1.3 + 2.3^2 + 3.3^3 + ... + n.3^n$

$$= \frac{(2n-1)3^{n+1}+3}{4} \qquad ...(i)$$

Step I: Then P(1) is the statement putting $n = 1$ in (*i*)

$P(1) : 1.3 = \frac{(2\times1-1)3^{1+1}+3}{4}$ or $3 = \frac{9+3}{4}$

or $3 = 3$ which is true. Thus, P(1) is true.

Step II: Assume that P(k) is true for some natural number k.

i.e., let $1.3 + 2.3^2 + 3.3^3 + ... + k.3^k = \frac{(2k-1)3^{k+1}+3}{4}$ ...(*ii*)

Step III: We want to prove that [changing k to $k + 1$ in (*ii*)]

$P(k + 1) : 1.3 + 2.3^2 + 3.3^3 + ... + k.3^k + (k + 1).3^{k+1}$

$$= \frac{[2(k+1)-1].3^{k+2}+3}{4}$$

$$= \frac{(2k+1).3^{k+2}+3}{4} \text{ is also true.}$$

Now, L.H.S. of P $(k + 1) = 1.3 + 2.3^2 + 3.3^3 + ... + k.3^k + (k + 1).3^{k+1}$

$$= \frac{(2k-1).3^{k+1}+3}{4} + (k+1).3^{k+1} \qquad \text{[Using (ii)]}$$

$$= \frac{(2k-1).3^{k+1}+3+4(k+1).3^{k+1}}{4}$$

$$= \frac{(2k-1+4k+4).3^{k+1}+3}{4}$$

$$= \frac{(6k+3).3^{k+1}+3}{4} = \frac{3(2k+1).3^{k+1}+3}{4}$$

$$= \frac{(2k+1).3^{k+2}+3}{4} = \text{R.H.S. of P}(k+1)$$

$\Rightarrow$ P$(k + 1)$ is true when P(k) is true.

Hence by P.M.I., P(n) is true for every positive integer n.

6. $1.2 + 2.3 + 3.4 + ... + n(n + 1) = \left[\frac{n(n+1)(n+2)}{3}\right]$.

Sol. Let P(n) : $1.2 + 2.3 + 3.4 + ... + n(n + 1)$

$$= \frac{n(n+1)(n+2)}{3} \qquad ...(i)$$

Step I: Then P(1) is the statement [Putting $n = 1$ in (*i*)]

P(1) : $1.2 = \frac{1\times2\times3}{3}$ or $2 = 2$ which is true.

Thus, P(1) is true.

Step II: Assume that P(k) is true for some natural number k.

i.e., let $1.2 + 2.3 + 3.4 + \ldots + k(k + 1) = \dfrac{k(k+1)(k+2)}{3}$ ...(*ii*)

We want to prove that [changing k to $(k + 1)$ in (*ii*)]

$P(k + 1) : 1.2 + 2.3 + 3.4 + \ldots + k(k + 1) + (k + 1)(k + 2)$

$= \dfrac{(k+1)(k+2)(k+3)}{3}$ is also true.

Now, L.H.S. of P $(k + 1) = 1.2 + 2.3 + 3.4 + \ldots + k(k + 1) + (k + 1)(k + 2)$

$= \dfrac{k(k+1)(k+2)}{3} + (k + 1)(k + 2)$ [Using (*ii*)]

$= (k + 1)(k + 2)\left(\dfrac{k}{3}+1\right)$

$= \dfrac{(k+1)(k+2)(k+3)}{3}$ = R.H.S. of P $(k + 1)$

$\Rightarrow$ P(k + 1) is true when P(k) is true. Hence by P.M.I., P(n) is true for every positive integer n.

7. $1.3 + 3.5 + 5.7 + \ldots + (2n - 1)(2n + 1) = \dfrac{n(4n^2+6n-1)}{3}$.

Sol. Let $P(n)$ be the statement

"$1.3 + 3.5 + 5.7 + \ldots + (2n - 1)(2n + 1) = \dfrac{n(4n^2+6n-1)}{3}$" ...(*i*)

Step I: Then P(1) is the statement [Putting $n = 1$ in (*i*)]

"$1.3 = \dfrac{1\times(4.1^2+6.1-1)}{3}$, *i.e.*, 3 = 3" which is true.

$\therefore$ P(1) is true.

Step II: Suppose P(k) is true, [Putting $n = k$ in (*i*)]

i.e., let $1.3 + 3.5 + \ldots + (2k - 1)(2k + 1) = \dfrac{k(4k^2+6k-1)}{3}$ ...(*ii*)

Step III: Putting $n = k + 1$ in (*i*):

P(k + 1) is the statement

"$1.3 + 3.5 + \ldots + (2k - 1)(2k + 1) + (2k + 1)(2k + 3)$"

$= \dfrac{(k+1)[4(k+1)^2+6(k+1)-1]}{3}$

$= \dfrac{(k+1)(4k^2+14k+9)}{3}$ $\left[\because (k+1)^2 = k^2+2k+1\right]$

We have to prove it.

Putting the value of L.H.S. of eqn. (*ii*) in L.H.S. of P(k + 1), we have 1.3 + 3.5 + ... + ($2k$ – 1) ($2k$ + 1) + ($2k$ + 1) ($2k$ + 3)

$$= \frac{k(4k^2+6k-1)}{3} + (2k+1)(2k+3)$$

$$= \frac{1}{3}[(4k^3+6k^2-k)+3(4k^2+8k+3)]$$

$$= \frac{1}{3}(4k^3+18k^2+23k+9) \qquad \text{...(iii)}$$

Breaking terms to get (k + 1) as a factor as required in R.H.S. of P(k + 1)

$$= \frac{1}{3}(4k^3+4k^2+14k^2+14k+9k+9) \text{ Note}$$

$$= \frac{1}{3}[4k^2(k+1)+14k(k+1)+9(k+1)]$$

$$= \frac{(k+1)(4k^2+14k+9)}{3}$$

$$= \frac{(k+1)[4k^2+8k+4+6k+6-1]}{3}$$

$$= \frac{(k+1)[4(k^2+2k+1)+6(k+1)-1]}{3}$$

$$= \frac{(k+1)[4(k+1)^2+6(k+1)-1]}{3}$$

$\Rightarrow$ P(k + 1) is true.

$\therefore$ By the principle of mathematical induction, P(n) is true for all natural numbers n.

Remark: After arriving at Eqn (*iii*) in the above solution, we can also proceed as below.

$$\text{R.H.S. of } P(k+1) = \frac{(k+1)[4k^2+14k+9)}{3}$$

$$= \frac{1}{3}(4k^3+14k^2+9k+4k^2+14k+9)$$

$$= \frac{1}{3}(4k^3+18k^2+23k+9) \qquad \text{...(iv)}$$

From (*iii*) and (*iv*) L.H.S. of P (k + 1) = R.H.S. of P (k + 1)

$\therefore$ P(k + 1) is true.

$\therefore$ By the principle of mathematical induction, P (n) is true for all natural numbers n.

8. $\mathbf{1.2 + 2.2^2 + 3.2^3 + ... + n.2^n = (n - 1)\ 2^{n+1} + 2}$

Sol. Let P(n) : $1.2 + 2.2^2 + 3.2^3 + ... + n.2^n = (n - 1)\ 2^{n+1} + 2$ ...(i)

Step I: Therefore P (1) is the statement [Putting $n = 1$ in (i)]

P(1) : $1.2 = 0 + 2$ or $2 = 2$ which is true.

Thus, P(1) is true.

Step II: Assume that P(k) is true for some natural number k.

i.e., let $1.2 + 2.2^2 + 3.2^3 + ... + k.2^k = (k - 1).2^{k+1} + 2$ ...(ii)

Step III: We want to prove that

P(k + 1) : $1.2 + 2.2^2 + 3.2^3 + ... + k.2^k + (k + 1).2^{k+1}$

$= k.2^{k+2} + 2$ is also true.

Now, L.H.S. of P (k + 1) $= 1.2 + 2.2^2 + 3.2^3 + ... + k.2^k + (k + 1).2^{k+1}$

$= (k - 1).2^{k+1} + 2 + (k + 1).2^{k+1}$ [Using (ii)]

$= (k - 1 + k + 1).2^{k+1} + 2$

$= 2k.2^{k+1} + 2 = k.2^{k+2} + 2$ = R.H.S. of P (k + 1)

$\Rightarrow$ P(k + 1) is true when P(k) is true. Hence by P.M.I., P(n) is true for every positive integer n.

9. $\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + ... + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$

Sol. Let P (n) be the statement

$$\text{“}\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^n} = 1 - \frac{1}{2^n}\text{”} \quad ...(i)$$

Step I: Putting $n = 1$ in (i),

P (1) is the statement “$\dfrac{1}{2} = 1 - \dfrac{1}{2^1}$” which is true.

$\therefore$ P (1) is true.

Step II: Suppose P (k) is true, [Putting $n = k$ in (i)]

i.e., let

$$\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^k}=1-\frac{1}{2^k} \quad ...(ii)$$

Step III. Putting $n = k + 1$ in (*i*), P $(k + 1)$ is the statement

"$\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^{k+1}}$".

We have to prove it.

Putting the value of L.H.S. of eqn. (*ii*) in L.H.S. of P$(k + 1)$, we have

$$\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^k}+\frac{1}{2^{k+1}}=\left(1-\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}$$

$$=1-\frac{1}{(2^k)}+\frac{1}{(2^k 2^1)}$$

$$=1-\frac{1}{(2^k)}\left(1-\frac{1}{2}\right)=1-\frac{1}{(2^k)}\cdot\frac{1}{2}$$

$$=1-\frac{1}{(2^{k+1})}=\text{R.H.S. of P}(k+1)$$

$\Rightarrow$ P $(k + 1)$ is true.

$\therefore$ By the principle of mathematical induction, P (n) is true for all natural numbers n.

10. $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+...+\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4}$.

Sol. Let $P(n): \frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+...+\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4}$ ...(*i*)

Step I: Therefore P (1) is the statement [Putting n=1 in...(*i*)]

$P(1): \frac{1}{2.5}=\frac{1}{6\times1+4}$ or $\frac{1}{10}=\frac{1}{10}$ which is true.

Thus, P(1) is true.

Step II: Assume that P(k) is true for some natural number k.

i.e., let $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+...+\frac{1}{(3k-1)(3k+2)}=\frac{k}{6k+4}$ ...(*ii*)

Step III: Changing k to $(k+1)$ in (*ii*), P(k+1) is the statement.

$$P(k+1) : \frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \ldots + \frac{1}{(3k-1)(3k+2)} + \frac{1}{(3k+2)(3k+5)} = \frac{k+1}{6k+10}$$

Let us prove that P(*k* + 1) is true.

Now, L.H.S. of $P(k+1) = \frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \ldots +$

$$\frac{1}{(3k-1)(3k+2)} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)} \qquad \text{[Using } (ii)\text{]}$$

$$= \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{1}{3k+2}\left[\frac{k}{2} + \frac{1}{3k+5}\right] = \frac{1}{3k+2} \cdot \frac{k(3k+5)+2}{2(3k+5)}$$

$$= \frac{3k^2+5k+2}{(3k+2)(6k+10)} = \frac{3k^2+3k+2k+2}{(3k+2)(6k+10)}$$

$$= \frac{(k+1)(3k+2)}{(3k+2)(6k+10)}$$

$$= \frac{k+1}{6k+10} = \text{R.H.S. of } P(k+1)$$

$\Rightarrow$ P(*k* + 1) is true when P(*k*) is true. Hence by P.M.I., P(*n*) is true for every positive integer *n*.

11. $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$.

Sol. Let $P(n) : \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots + \frac{1}{n(n+1)(n+2)}$

$$= \frac{n(n+3)}{4(n+1)(n+2)} \qquad \ldots(i)$$

Step I: Therefore P(1) is the statement [Putting n=1 in (i)]

$$P(1) : \frac{1}{1.2.3} = \frac{1\times 4}{4\times 2\times 3} \quad \text{or} \quad \frac{1}{6} = \frac{1}{6} \text{ which is true.}$$

Thus, P(1) is true.

Step II: Assume that P(k) is true for some natural number k *i.e.*, let

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots + \frac{1}{k(k+1)(k+2)}$$

$$= \frac{k(k+3)}{4(k+1)(k+2)} \qquad \ldots(ii)$$

Step III: We want to prove that [changing k to (k + 1) in (ii)]

$$P(k+1) : \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

is also true.

Now, L.H.S of P(k + 1) = $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$

$$= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \quad \text{[Using (ii)]}$$

$$= \frac{1}{(k+1)(k+2)}\left[\frac{k(k+3)}{4} + \frac{1}{k+3}\right]$$

$$= \frac{1}{(k+1)(k+2)} \cdot \frac{k(k+3)^2+4}{4(k+3)}$$

$$= \frac{k(k^2+6k+9)+4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{k^3 + 6k^2 + 9k + 4}{4(k+1)(k+2)(k+3)} \qquad ...(iii)$$

Splitting Numerator to get $(k + 1)$ as a factor which is present in the Denominator here but absent in the Denominator of required R.H.S.

$$= \frac{k^3 + k^2 + 5k^2 + 5k + 4k + 4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{k^2(k+1) + 5k(k+1) + 4(k+1)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)(k^2 + 5k + 4)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)(k+4)}{4(k+2)(k+3)} \quad [\because k^2 + 5k + 4 = (k+1)(k+4)]$$

= R.H.S. of P $(k + 1)$

$\Rightarrow$ P$(k + 1)$ is true when P(k) is true. Hence by P.M.I., P(n) is true for every positive integer n.

Remark: After arriving at Eqn (iii) in the above solution, we can also proceed as below:

$$\text{R.H.S. of } P(k+1) = \frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

Multiplying both Numerator and denominator by $(k + 1)$ which is present as a factor in the denominator of (iii) but is absent in the denominator of the R.H.S. (above).

$$= \frac{(k+1)(k+1)(k+4)}{4(k+1)(k+2)(k+3)} = \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k^2+2k+1)(k+4)}{4(k+1)(k+2)(k+3)} = \frac{k^3+4k^2+2k^2+8k+k+4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)} \qquad ...(iv)$$

From (iii) and (iv) L.H.S. of P$(k + 1)$ = R.H.S. of P$(k + 1)$

$\therefore$ P$(k + 1)$ is true when P (k) is true.

Hence by P.M.I., P(n) is true for every positive integer n.

12. $a + ar + ar^2 + ... + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$.

Sol. Let P(n) : $a + ar + ar^2 + ... + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$ ...(i)

Step I: Therefore P (1) is the statement [Putting n=1 in (i)]

$$P(1) = a = \frac{a(r-1)}{r-1} \quad \text{or} \quad a = a \text{ which is true.}$$

Thus, P(1) is true.

Step II: Assume that P(k) is true for some natural number k.

i.e., let $a + ar + ar^2 + ... + ar^{k-1} = \dfrac{a(r^k - 1)}{r - 1}$ ...(ii)

Step III We want to prove that [changing k to $k + 1$ in (ii)]

P(k + 1) : $a + ar + ar^2 + ... + ar^{k-1} + ar^k$

$$= \frac{a(r^{k+1} - 1)}{r - 1}$$

is also true.

Now, L.H.S. of P (k + 1) = $a + ar + ar^2 + ... + ar^{k-1} + ar^k$

$$= \frac{a(r^k - 1)}{r - 1} + ar^k \qquad \text{[Using (ii)]}$$

$$= \frac{a(r^k - 1) + ar^k(r - 1)}{r - 1}$$

$$= \frac{a(r^k - 1 + r^{k+1} - r^k)}{r - 1}$$

$$= \frac{a(r^{k+1} - 1)}{r - 1} = \text{R.H.S. of P } (k + 1)$$

$\Rightarrow$ P(k + 1) is true when P(k) is true. Hence by P.M.I., P(n) is true for every positive integer n.

13. $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)...\left(1+\frac{2n+1}{n^2}\right)=(n+1)^2.$

Sol. Let P (n) be the statement

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)...\left(1+\frac{2n+1}{n^2}\right)=(n+1)^2 \qquad ...(i)$$

Step I: Putting $n = 1$ in (i),

P(1) is the statement $\left(1+\frac{3}{1}\right)=(1+1)^2$

or $4 = 2^2$ or $4 = 4$ which is true.

Thus P (1) is true.

Step II: Let P(k) be true *i.e.,* P(n) be true for $n = k$.

Putting $n = k$ in (i), we have

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)...\left(1+\frac{2k+1}{k^2}\right)=(k+1)^2 \qquad ...(ii)$$

Step III: Putting $n = k + 1$ in (i), [on changing k to (k + 1) in (ii)] P(k + 1) is the statement

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)...\left(1+\frac{2k+1}{k^2}\right)\left[1+\frac{2(k+1)+1}{(k+1)^2}\right]$$
$$=(k+1+1)^2$$

i.e., $$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)...\left(1+\frac{2k+1}{k^2}\right)\left(1+\frac{2k+3}{(k+1)^2}\right)$$
$$=(k+2)^2$$

We have to prove it.

Putting the value of L.H.S. of eqn. (ii) in L.H.S. of P(k + 1), we have

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)...\left(1+\frac{2k+1}{k^2}\right)\left(1+\frac{2k+3}{(k+1)^2}\right)$$

$$= (k+1)^2 \left[1 + \frac{2k+3}{(k+1)^2}\right]$$

$$= (k+1)^2 \left[\frac{(k+1)^2 + 2k + 3}{(k+1)^2}\right]$$

$$= k^2 + 1 + 2k + 2k + 3 = k^2 + 4k + 4 = (k+2)^2$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(n) is true for all natural numbers n by P.M.I.

14. $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right) \ldots \left(1+\frac{1}{n}\right) = n + 1.$

Sol. Let P(n) : $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right) \ldots \left(1+\frac{1}{n}\right) = n + 1$ ...(i)

Step I: Therefore P (1) is the statement [Putting n = 1 in (i)]

P(1) : $1 + \frac{1}{1} = 1 + 1$ or $2 = 2$ which is true.

Thus, P(1) is true.

Step II: Assume that P(k) is true for some natural number k.

[Putting $n = k$ in (i)]

i.e., let $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right) \ldots \left(1+\frac{1}{k}\right) = k + 1$ ...(ii)

Step III: We want to prove that [changing k to k + 1 in (ii)]

P(k + 1) : $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right) \ldots \left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right)$
$= k + 2$

is also true.

Now, L.H.S. of P (k + 1)

$$= \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right) \ldots \left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right)$$

$$= (k+1)\left(1+\frac{1}{k+1}\right)$$ [Using (ii)]

$$= (k+1)\left(\frac{k+1+1}{k+1}\right) = k+2$$

$\Rightarrow$ P(k + 1) is true when P(k) is true. Hence by P.M.I., P(n) is true for every positive integer n.

15. $1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$

Sol. Let P(n) be the statement

$$\text{“}1^2 + 3^2 + 5^2 ... + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}\text{”}. \quad ...(i)$$

Step I: Then P(1) is the statement [Putting n = 1 in (i)]

$$\text{“}1^2 = \frac{1\times(2\times1-1)(2\times1+1)}{3}, \text{ i.e., } 1 = 1\text{” which is true.}$$

$\therefore$ P(1) is true.

Step II: Suppose P(k) is true, [Putting $n = k$ in (i)]

i.e., let

$$1^2 + 3^2 + 5^2 + ... + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3} \quad ...(ii)$$

Step III: P(k + 1) is the statement [Putting $n = k + 1$ in (i)]

$$\text{“}1^2 + 3^2 + ... + (2k-1)^2 + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}\text{”}.$$

We have to prove it.

Putting the value of L.H.S. of eqn. (ii) in L.H.S. of P(k + 1), we have

$$1^2 + 3^2 + ... + (2k-1)^2 + (2k+1)^2 = \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

Now taking (2k + 1) common from R.H.S. and taking 3 as L.C.M. on R.H.S.;

$$= \frac{2k+1}{3}[k(2k-1) + 3(2k+1)]$$

$$= \frac{2k+1}{3}(2k^2 + 5k + 3)$$

$$= \frac{2k+1}{3}(2k^2 + 2k + 3k + 3)$$

$$= \frac{2k+1}{3}[2k(k+1)+3(k+1)]$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3}$$

$\Rightarrow$ P(k + 1) is true.

$\therefore$ By the principle of mathematical induction, P(n) is true for all natural numbers n.

16. $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$.

Sol. Let P(n) : $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}$

$$= \frac{n}{3n+1} \qquad \ldots(i)$$

Step I: Therefore P (1) is the statement [Putting n = 1 in (i)]

P(1) = $\frac{1}{1.4}=\frac{1}{3\times1+1}$ or $\frac{1}{4}=\frac{1}{4}$ which is true.

Thus, P(1) is true.

Step II: Assume that P(k) is true for some natural number k. [Putting n = k in (i)]

$i.e.$, let $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}$

$$= \frac{k}{3k+1} \qquad \ldots(ii)$$

Step III: We want to prove that [changing k to k + 1 in (ii)]

P(k + 1) : $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots$

$$+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3k+1)(3k+4)}=\frac{k+1}{3k+4}$$

is also true.

Now, L.H.S. of P(*k* + 1) = $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3k+1)(3k+4)}$

$$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$$ [Using (*ii*)]

$$= \frac{1}{3k+1} \cdot \left[k + \frac{1}{3k+4}\right]$$

$$= \frac{1}{3k+1} \cdot \frac{3k^2+4k+1}{3k+4}$$

$\because$ $3k^2 + 4k + 1$
$= 3k^2 + 3k + k + 1$
$= 3k(k + 1) + 1(k + 1)$
$= (3k + 1)(k + 1)$

$$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)} = \frac{k+1}{3k+4}$$ = R.H.S. of P(*k* + 1)

$\Rightarrow$ P(*k* + 1) is true when P(*k*) is true. Hence by P.M.I., P(*n*) is true for every positive integer *n*.

17. $\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \ldots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$.

Sol. Let P(*n*) : $\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \ldots + \frac{1}{(2n+1)(2n+3)}$

$$= \frac{n}{3(2n+3)} \qquad \ldots(i)$$

Step I: Putting $n = 1$ in (*i*),

P(1) : $\frac{1}{3.5} = \frac{1}{3(2\times1+3)}$ or $\frac{1}{15} = \frac{1}{15}$ which is true.

Thus, P(1) is true.

Step II: Assume that P(*k*) is true for some natural number *k*.

i.e., let $\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \ldots + \frac{1}{(2k+1)(2k+3)}$

$$= \frac{k}{3(2k+3)} \qquad \ldots(ii)$$

Step III: We want to prove that [changing *k* to (*k* +1) in (*ii*)]

$$P(k+1) : \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \ldots + \frac{1}{(2k+1)(2k+3)} + \frac{1}{(2k+3)(2k+5)} = \frac{k+1}{3(2k+5)}$$

is also true.

Now, L.H.S. of $P(k+1) = \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \ldots + \frac{1}{(2k+1)(2k+3)} + \frac{1}{(2k+3)(2k+5)}$

$$= \frac{k}{3(2k+3)} + \frac{1}{(2k+3)(2k+5)} \quad \text{[Using } (ii)\text{]}$$

$$= \frac{1}{2k+3}\left[\frac{k}{3} + \frac{1}{2k+5}\right]$$

$$= \frac{1}{2k+3} \cdot \frac{2k^2+5k+3}{3(2k+5)}$$

$$= \frac{1}{2k+3} \cdot \frac{(2k+3)(k+1)}{3(2k+5)}$$

$\because\ 2k^2 + 5k + 3 = 2k^2 + 2k + 3k + 3 = 2k(k+1) + 3(k+1) = (2k+3)(k+1)$

$$= \frac{k+1}{3(2k+5)} = \text{R.H.S. of } P(k+1)$$

$\Rightarrow$ $P(k + 1)$ is true when $P(k)$ is true. Hence by P.M.I., $P(n)$ is true for every positive integer n.

18. $1 + 2 + 3 + \ldots + n < \frac{1}{8}(2n + 1)^2$.

Sol. Let $P(n) : 1 + 2 + 3 + \ldots + n < \frac{1}{8}(2n + 1)^2$ ...(*i*)

Step I: Putting $n = 1$ in (*i*),

$P(1) : 1 < \frac{1}{8}(2 \times 1 + 1)^2$ or $1 < \frac{9}{8}$ which is true.

Thus, $P(1)$ is true.

Step II: Assume that $P(k)$ is true for some natural number k. Putting $n = k$ in (*i*)

i.e., let $1 + 2 + 3 + \ldots + k < \frac{1}{8}(2k + 1)^2$ ...(*ii*)

Step III: We want to prove that [changing k to $(k + 1)$ in (ii)]

$$P(k + 1) : 1 + 2 + 3 + ... + k + (k + 1) < \frac{1}{8} (2k + 3)^2$$

Now,

$$1 + 2 + 3 + ... + k + (k + 1)$$

$$< \frac{1}{8} (2k + 1)^2 + (k + 1) \quad \text{[Using } (ii)\text{]}$$

$$= \frac{1}{8} [(2k + 1)^2 + 8(k + 1)]$$

$$= \frac{1}{8} [4k^2 + 4k + 1 + 8k + 8]$$

$$= \frac{1}{8} (4k^2 + 12k + 9)$$

$$= \frac{1}{8} (2k + 3)^2$$

$$[\because \ 4k^2 + 12k + 9 = 4k^2 + 6k + 6k + 9$$
$$= 2k (2k + 3) + 3 (2k + 3)$$
$$= (2k + 3) (2k + 3) = (2k + 3)^2]$$

$$\therefore \ 1 + 2 + 3 + ... + k + (k + 1) < \frac{1}{8}(2k + 3)^2$$

$\Rightarrow$ P$(k + 1)$ is true when P(k) is true. Hence by P.M.I., P(n) is true for every positive integer n.

19. $n(n + 1)(n + 5)$ is a multiple of 3.

Sol. Let P(n) : $n(n + 1)(n + 5)$ is a multiple of 3. ...(i)

Step I: Putting $n = 1$ in (i),

P(1) : 1(2)(6) is a multiple of 3 or 12 is a multiple of 3 which is true.

Thus, P(1) is true.

Step II: Assume that P(k) is true for some natural number k.

i.e., let $k(k + 1)(k + 5)$ be a multiple of 3

i.e., let $k(k^2 + 6k + 5)$ be a multiple of 3

i.e., $k^3 + 6k^2 + 5k = 3a$ where $a \in Z$

$\Rightarrow k^3 = 3a - 6k^2 - 5k$ ...(*ii*)

Step III: We want to prove that

[putting $n = k + 1$ in (*i*)]

P(k + 1) : $(k + 1)(k + 2)(k + 6)$ is a multiple of 3 is also true.

Now, $(k + 1)(k + 2)(k + 6) = (k + 1)(k^2 + 8k + 12)$

$= k^3 + 8k^2 + 12k + k^2 + 8k + 12 = k^3 + 9k^2 + 20k + 12$

$= (3a - 6k^2 - 5k) + 9k^2 + 20k + 12$

[By substituting for k^3 from (*ii*)]

$= 3a + 3k^2 + 15k + 12$

$= 3(a + k^2 + 5k + 4)$

$= 3b$ where $b = (a + k^2 + 5k + 4) \in Z$

$\Rightarrow$ $(k + 1)(k + 2)(k + 6)$ is a multiple of 3.

$\Rightarrow$ P(k + 1) is true when P(k) is true. Hence by P.M.I., P(n) is true for every positive integer n.

20. $10^{2n-1} + 1$ is divisible by 11.

Sol. Let P(n) : $10^{2n-1} + 1$ is divisible by 11 ...(*i*)

Step I: Putting $n = 1$ in (*i*),

P(1) : $10^1 + 1$ is divisible by 11 or 11 is divisible by 11, which is true. Thus, P(1) is true.

Step II: Assume that P(k) is true for some natural number k.

Putting $n = k$ in (*i*),

i.e., let $10^{2k-1} + 1$ be divisible by 11

i.e., $10^{2k-1} + 1 = 11a$ where $a \in Z$.

$\Rightarrow 10^{2k-1} = 11a - 1$ ...(*ii*)

Step III: We want to prove that

Putting $n = (k + 1)$ in (*i*),

P(k + 1) : $10^{2(k+1)-1} + 1 = 10^{2k+1} + 1$ is divisible by 11, is also true.

Now,

$10^{2k+1} + 1 = 10^{(2k-1)+2} + 1$

$= 10^{2k-1} \cdot 10^2 + 1$

$= 100(11a - 1) + 1$ [Using (*ii*)]

$= 1100a - 99 = 11(100a - 9)$

$= 11b$ where $b = (100a - 9) \in Z$

$\Rightarrow$ $10^{2k+1} + 1$ is divisible by 11.

$\Rightarrow$ P(k + 1) is true when P(k) is true. Hence by P.M.I., P(n) is true for every positive integer n.

21. $x^{2n} - y^{2n}$ **is divisible by** $x + y$**.**

Sol. Let P(n) : $x^{2n} - y^{2n}$ is divisible by $x + y$. ...(*i*)

Step I: Putting $n = 1$ in (*i*),

P(1) : $x^2 - y^2$ is divisible by $x + y$

or $(x + y)(x - y)$ is divisible by $x + y$, which is true.

Thus, P(1) is true.

Step II: Assume that P(k) is true for some natural number k.

i.e., let $x^{2k} - y^{2k}$ be divisible by $x + y$

i.e., $x^{2k} - y^{2k} = a(x + y)$ where $a \in Z$

$\Rightarrow x^{2k} = a\,(x + y) + y^{2k}$ ...(*ii*)

Step III: We want to prove that

P(k + 1) : $x^{2k+2} - y^{2k+2}$ is divisible by $x + y$, is also true.

Now, $x^{2k+2} - y^{2k+2}$

$= x^2 \cdot x^{2k} - y^2 \cdot y^{2k}$

$= x^2\,[a(x + y) + y^{2k}] - y^2 \cdot y^{2k}$ [Using (*ii*)]

$= ax^2\,(x + y) + x^2 \cdot y^{2k} - y^2 \cdot y^{2k}$

$= ax^2\,(x + y) + y^{2k}\,(x^2 - y^2)$

$= ax^2\,(x + y) + y^{2k}\,(x + y)\,(x - y)$

$= (x + y)\,[ax^2 + y^{2k}\,(x - y)]$

$= b(x + y)$ where $b = [ax^2 + y^{2k}\,(x - y)] \in Z$

$\Rightarrow$ $x^{2k+2} - y^{2k+2}$ is divisible by $x + y$

$\Rightarrow$ P(k + 1) is true when P(k) is true. Hence by P.M.I., P(n) is true for every positive integer n.

22. $3^{2n+2} - 8n - 9$ **is divisible by 8.**

Sol. Let P(n) : $3^{2n+2} - 8n - 9$ is divisible by 8. ...(*i*)

Step I : Putting $n = 1$ in (*i*)

P(1) : $3^4 - 8 - 9 = 81–17$ is divisible by 8

or 64 is divisible by 8, which is true.

Thus, P(1) is true.

Step II: Assume that P(k) is true for some natural number k.

i.e., let $3^{2k+2} - 8k - 9$ be divisible by 8

i.e., $3^{2k+2} - 8k - 9 = 8a$ where $a \in Z$

i.e., $3^{2k+2} = 8a + 8k + 9$ ... (*ii*)

Step III: We want to prove that [Putting $n = k + 1$ in (*i*)]

P(k + 1) : $3^{2(k+1)+2} - 8(k+1) - 9$ is divisible by 8, is also true.

Now, $3^{2k+4} - 8(k+1) - 9$

$= 3^{(2k+4)} - 8k - 8 - 9$

$= 3^{(2k+2)+2} - 8k - 8 - 9$

$= 3^{2k+2} . 3^2 - 8k - 17$

$= 9(8a + 8k + 9) - 8k - 17$ [Using (*ii*)]

$= 72a + 72k + 81 - 8k - 17$

$= 72a + 64k + 64 = 8(9a + 8k + 8)$

$= 8b$, where $b = 9a + 8k + 8 \in Z$

$\Rightarrow$ $3^{2k+4} - 8(k+1) - 9$ is divisible by 8.

$\Rightarrow$ P(k + 1) is true when P(k) is true. Hence, by P.M.I., P(n) is true for every positive integer n.

23. $41^n - 14^n$ is a multiple of 27.

Sol. Let P(n) : $41^n - 14^n$ is a multiple of 27. ...(*i*)

Step I: Putting $n = 1$ in (*i*),

P(1) : 41 – 14 is a multiple of 27

or 27 is a multiple of 27, which is true.

Thus, P(1) is true.

Step II: Assume that P(k) is true for some natural number k.

Putting $n = k$ in (*i*)

i.e., let $41^k - 14^k$ be a multiple of 27

i.e., $41^k - 14^k = 27a$, where $a \in Z$

$\Rightarrow 41^k = 27a + 14^k$ ...(*ii*)

Step III: We want to prove that [Putting $n = k + 1$ in (i)]

$P(k + 1) : 41^{k+1} - 14^{k+1}$ is a multiple of 27 is also true.

Now, $41^{k+1} - 14^{k+1} = 41 \,.\, 41^k - 14 \,.\, 14^k$

$= 41(27a + 14^k) - 14 \,.\, 14^k$ [Using (ii)]

$= 41 \times 27a + 41 \,.\, 14^k - 14 \,.\, 14^k$

$= 41 \times 27a + 27 \,.\, 14^k = 27(41a + 14^k)$

$= 27b$, where $b = (41a + 14^k) \in Z$

$\Rightarrow$ $41^{k+1} - 14^{k+1}$ is a multiple of 27.

$\Rightarrow$ $P(k + 1)$ is true when $P(k)$ is true. Hence by P.M.I., $P(n)$ is true for every positive integer n.

24. $\mathbf{2n + 7 < (n + 3)^2}$.

Sol. Let $P(n)$ be the statement "$2n + 7 < (n + 3)^2$"... (i)

Step I: Then $P(1)$ is the statement "$2 \times 1 + 7 < (1 + 3)^2$ or $9 < 16$" which is true. Thus $P(1)$ is true.

Step II: Suppose $P(k)$ is true, [Putting $n = k$ in (i)],

then $2k + 7 < (k + 3)^2$ ...(ii)

Step III: $P(k + 1)$ is the statement "$2(k + 1) + 7 < (k + 4)^2$

$= k^2 + 8k + 16$"

Now, $2(k + 1) + 7 = (2k + 7) + 2$

$< (k + 3)^2 + 2$ [Using (ii)]

$= k^2 + 9 + 6k + 2 = k^2 + 6k + 11$

Adding and subtracting $(2k + 5)$ in the R.H.S.;

$= k^2 + 6k + 11 + 2k + 5 - (2k + 5)$

$= (k^2 + 8k + 16) - (2k + 5)$

$= (k + 4)^2 - (2k + 5)$

$< (k + 4)^2$, since $2k + 5 > 0$ for all $k \in N$

$\Rightarrow$ $P(k + 1)$ is true.

$\therefore$ By the principle of mathematical induction, $P(n)$ is true for all $n \in N$.