

5 Complex Numbers and Quadratic Equations

Lesson at a Glance

1. A number of the form $a + ib$, where a and b are real numbers and $i = \sqrt{-1}$, is called a complex number and $a + ib$ is called **standard form** of complex number.
2. If $z = a + ib$ then $Re(z) = a$ and $Im(z) = b$, **the coefficient of i**
3. $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $\frac{1}{i} = -i$.
4. For any integer k , $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$.
5. (i) For any two real numbers a and b , $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only if at least one of a and b is either 0 or positive.
(ii) and $\sqrt{a} \sqrt{b} = -\sqrt{ab}$ if both a and b are real negative numbers.
6. Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are **equal** if and only if $a = c$ and $b = d$ i.e., $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.
We call it as "Equating real and imaginary parts" on both sides.
7. The conjugate of the complex number $z = a + ib$ is denoted by \bar{z} and is given by $\bar{z} = a - ib$.
8. To express $\frac{z_1}{z_2}$, $z_2 \neq 0$, in the standard form $a + ib$, multiply and divide by \bar{z}_2 , the conjugate of denominator.
9. The modulus of the complex number $z = a + ib$ is denoted by $|z|$ and is defined as $|z| = \sqrt{a^2 + b^2}$. and hence $|z|$ is never negative.

10. If z, z_1 and z_2 are complex numbers,

(i) $\overline{(\bar{z})} = z$ (ii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(iii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ (iv) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(v) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

(vi) $|z| = 0$ if and only if $z = 0$

(vii) $|z| = |\bar{z}| = |-z|$

(viii) $z\bar{z} = |z|^2$

(ix) $|z^2| = |z|^2$

(x) $|z_1 + z_2| \leq |z_1| + |z_2|$. (Triangle inequality)

11. Multiplicative inverse (or Reciprocal) of non-zero complex number z i.e.,

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{|z|^2}. \quad (\because z\bar{z} = |z|^2)$$

12. Every non-zero complex number z can be written as

$$z = x + iy = r(\cos \theta + i \sin \theta)$$

$$(\because x = r \cos \theta \text{ and } y = r \sin \theta.)$$

This is called **polar form** of z where $r = |z|$, the modulus of z and $\theta = \arg z$, where $-\pi < \theta \leq \pi$.

This angle θ is always expressed in radians.

13. If θ is the **argument** of a complex number $z = x + iy$ (as

defined in 12), then θ is given by $\tan \theta = \frac{y}{x}$

θ mentioned in all the four cases (or as in the adjoining figure) listed

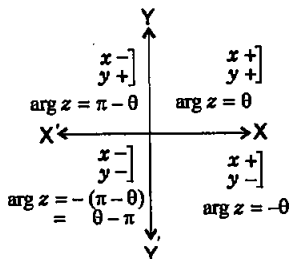
below is s.t. $0 \leq \theta < \frac{\pi}{2}$.

(i) If (x, y) lies in the first quadrant, then $\arg z = \theta$

(ii) If (x, y) lies in the second quadrant, then $\arg z = \pi - \theta$

(iii) If (x, y) lies in third quadrant, then $\arg z = -(\pi - \theta) = \theta - \pi$.

(iv) If (x, y) lies in fourth quadrant, then $\arg z = -\theta$



14. Every real number is also a complex number. ($\because x = x + 0i$)
15. A complex number of the form $z = iy$ ($y \neq 0, y \in R$) is called a **purely imaginary number**. (\Rightarrow Real part of $z = 0$)
16. Every quadratic equation has two complex roots.

TEXTBOOK QUESTIONS SOLVED

EXERCISE 5.1 (Page No.: 103–104)

Express each of the complex numbers given in Exercises 1 to 10 in the form $a + ib$:

1. $(5i) \left(-\frac{3}{5}i\right)$.

Sol. $(5i) \left(-\frac{3}{5}i\right) = (5) \left(-\frac{3}{5}\right) i^2 = -3(-1) = 3$ [$\because i^2 = -1$]
 $= 3 + i0$.

2. $i^9 + i^{19}$.

Sol. $i^9 + i^{19} = i^8 \cdot i + i^{16} \cdot i^3 = (i^4)^2 i + (i^4)^4 i^3 = (1)^2 i + (1)^4 \cdot (-i)$
 $= i - i = 0 = 0 + 0i$

3. i^{-39}

Sol. $i^{-39} = \frac{1}{i^{39}} = \frac{1}{i^{36} \cdot i^3} = \frac{1}{(i^4)^9 \cdot i^3}$
 $= \frac{1}{(1)^9 \cdot (-i)} = \frac{1}{-i}$ [$\because i^4 = 1$ and $i^3 = -i$]

Rationalising

$$= \frac{1}{-i} \times \frac{i}{i} = \frac{i}{-i^2} = \frac{i}{1}$$
 [$\because i^2 = -1$]
 $= i = 0 + i$.

4. $3(7 + i7) + i(7 + i7)$

Sol. $3(7 + i7) + i(7 + i7) = 21 + 21i + 7i + 7i^2$
 $= 21 + 28i - 7$ [$\because i^2 = -1$]
 $= 14 + 28i$.

5. $(1 - i) - (-1 + i6)$

Sol. $(1 - i) - (-1 + i6) = 1 - i + 1 - 6i = 2 - 7i$.

6. $\left(\frac{1}{5} + i\frac{2}{5}\right) - \left(4 + i\frac{5}{2}\right)$

Sol. $\left(\frac{1}{5} + i\frac{2}{5}\right) - \left(4 + i\frac{5}{2}\right) = \frac{1}{5} + \frac{2}{5}i - 4 - \frac{5}{2}i$
 $= \left(\frac{1}{5} - 4\right) + \left(\frac{2}{5} - \frac{5}{2}\right)i$
 $= \frac{1-20}{5} + \frac{4-25}{10}i$
 $= -\frac{19}{5} - \frac{21}{10}i.$

7. $\left[\left(\frac{1}{3} + i\frac{7}{3}\right) + \left(4 + i\frac{1}{3}\right)\right] - \left(-\frac{4}{3} + i\right)$

Sol. $\left[\left(\frac{1}{3} + i\frac{7}{3}\right) + \left(4 + i\frac{1}{3}\right)\right] - \left(-\frac{4}{3} + i\right)$
 $= \left(\frac{1}{3} + \frac{7}{3}i + 4 + \frac{1}{3}i\right) + \frac{4}{3} - i$
 $= \left[\left(\frac{1}{3} + 4\right) + \left(\frac{7}{3} + \frac{1}{3}\right)i\right] + \frac{4}{3} - i$
 $= \left(\frac{13}{3} + \frac{8}{3}i\right) + \frac{4}{3} - i = \left(\frac{13}{3} + \frac{4}{3}\right) + \left(\frac{8}{3} - 1\right)i$
 $= \frac{17}{3} + \frac{5}{3}i.$

8. $(1 - i)^4$

Sol. $(1 - i)^4 = [(1 - i)^2]^2 = (1 + i^2 - 2i)^2$
 $= (1 - 1 - 2i)^2 = (-2i)^2 = 4i^2$
 $= -4 = -4 + 0i.$

9. $\left(\frac{1}{3} + 3i\right)^3$

Sol. $\left(\frac{1}{3} + 3i\right)^3 = \left(\frac{1}{3}\right)^3 + (3i)^3 + 3 \cdot \frac{1}{3} \cdot 3i \left(\frac{1}{3} + 3i\right)$
 $[\because (a + b)^3 = a^3 + b^3 + 3ab(a + b)]$

$$\begin{aligned}
 &= \frac{1}{27} + 27i^3 + 3i \left(\frac{1}{3} + 3i \right) \\
 &= \frac{1}{27} - 27i + i + 9i^2 && [\because i^3 = -i] \\
 &= \frac{1}{27} - 26i - 9 && [\because i^2 = -1] \\
 &= \left(\frac{1}{27} - 9 \right) - 26i = \frac{1 - 243}{27} - 26i \\
 &= -\frac{242}{27} - 26i.
 \end{aligned}$$

10. $\left(-2 - \frac{1}{3}i \right)^3$

Sol. $\left(-2 - \frac{1}{3}i \right)^3 = \left[- \left(2 + \frac{1}{3}i \right) \right]^3 = - \left(2 + \frac{1}{3}i \right)^3$

$$\begin{aligned}
 &= - \left[2^3 + \left(\frac{1}{3}i \right)^3 + 3 \cdot 2 \cdot \frac{1}{3}i \left(2 + \frac{1}{3}i \right) \right] \\
 &= [\because (a + b)^3 = a^3 + b^3 + 3ab(a + b)] \\
 &= - \left[8 + \frac{1}{27}i^3 + 2i \left(2 + \frac{1}{3}i \right) \right] \\
 &= - \left[8 + \frac{1}{27}i^3 + 2i \left(2 + \frac{1}{3}i \right) \right] \\
 &= - \left[8 - \frac{1}{27}i + 4i + \frac{2}{3}i^2 \right] && [\because i^3 = -i] \\
 &= - \left(8 - \frac{1}{27}i + 4i - \frac{2}{3} \right) && [\because i^2 = -1] \\
 &= - \left[\left(8 - \frac{2}{3} \right) + \left(4 - \frac{1}{27} \right) i \right] \\
 &= - \left(\frac{22}{3} + \frac{107}{27}i \right) \\
 &= -\frac{22}{3} - \frac{107}{27}i.
 \end{aligned}$$

Find the multiplicative inverse of each of the complex numbers given in the Exercises 11 to 13.

11. $4 - 3i$

Sol. Multiplicative inverse of $z = 4 - 3i$ is

$$\begin{aligned} z^{-1} &= \frac{1}{z} = \frac{1}{4 - 3i} = \frac{1}{4 - 3i} \times \frac{4 + 3i}{4 + 3i} = \frac{4 + 3i}{4^2 - (3i)^2} \\ &= \frac{4 + 3i}{16 - 9i^2} = \frac{4 + 3i}{16 + 9} = \frac{4 + 3i}{25} \\ &= \frac{4}{25} + \frac{3}{25} i. \end{aligned}$$

Second Solution

Multiplicative inverse of $z = 4 - 3i$ is

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{4 + 3i}{(4)^2 + (-3)^2} = \frac{4 + 3i}{16 + 9} = \frac{4 + 3i}{25} = \frac{4}{25} + \frac{3}{25} i$$

[\because If $z = x + iy$, then $\bar{z} = x - iy$ and $|z| = \sqrt{x^2 + y^2}$ and hence $|z|^2 = x^2 + y^2$]

12. $\sqrt{5} + 3i$

Sol. Multiplicative inverse of $z = \sqrt{5} + 3i$ is

$$\begin{aligned} z^{-1} &= \frac{1}{z} = \frac{1}{\sqrt{5} + 3i} = \frac{1}{\sqrt{5} + 3i} \times \frac{\sqrt{5} - 3i}{\sqrt{5} - 3i} \\ &= \frac{\sqrt{5} - 3i}{(\sqrt{5})^2 - (3i)^2} = \frac{\sqrt{5} - 3i}{5 - 9i^2} = \frac{\sqrt{5} - 3i}{5 + 9} \\ &= \frac{\sqrt{5} - 3i}{14} = \frac{\sqrt{5}}{14} - \frac{3}{14} i. \end{aligned}$$

Second Solution

Multiplicative inverse of $z = \sqrt{5} + 3i$ is

$$\begin{aligned} z^{-1} &= \frac{\bar{z}}{|z|^2} = \frac{\sqrt{5} - 3i}{(\sqrt{5})^2 + (3)^2} = \frac{\sqrt{5} - 3i}{5 + 9 = 14} \\ &= \frac{\sqrt{5}}{14} - \frac{3i}{14}. \end{aligned}$$

13. $-i$

Sol. Multiplicative inverse of $z = -i$ is

$$z^{-1} = \frac{1}{z} = \frac{1}{-i} = \frac{1}{-i} \times \frac{i}{i} = \frac{i}{-i^2} = \frac{i}{-(-1)} = i = 0 + i.$$

Second Solution

Multiplicative inverse of $z = -i = 0 - i$ is

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{0+i}{(0)^2+(-1)^2} = \frac{0+i}{1} = 0 + i$$

14. Express the following expression in the form $a + ib$:

$$\frac{(3 + i\sqrt{5})(3 - i\sqrt{5})}{(\sqrt{3} + \sqrt{2}i) - (\sqrt{3} - i\sqrt{2})}$$

Sol.
$$\frac{(3 + i\sqrt{5})(3 - i\sqrt{5})}{(\sqrt{3} + \sqrt{2}i) - (\sqrt{3} - i\sqrt{2})}$$

Using $(a + b)(a - b) = a^2 - b^2$

$$= \frac{3^2 - (i\sqrt{5})^2}{\sqrt{3} + \sqrt{2}i - \sqrt{3} + \sqrt{2}i} = \frac{9 - 5i^2}{2\sqrt{2}i}$$

$$= \frac{9+5}{2\sqrt{2}i} = \frac{7}{\sqrt{2}i} \times \frac{-\sqrt{2}i}{-\sqrt{2}i} \quad (\text{Rationalising})$$

$$= \frac{-7\sqrt{2}i}{-2i^2} = \frac{-7\sqrt{2}i}{2} \quad [\because i^2 = -1]$$

$$= 0 - \frac{7\sqrt{2}}{2}i.$$

EXERCISE 5.2 (Page No.: 108)

Find the modulus and the arguments of each of the complex numbers in Exercises 1 to 2.

1. $z = -1 - i\sqrt{3}$

Sol. $z = -1 - i\sqrt{3} = x + iy$

is represented by the point P $(-1, -\sqrt{3})$ which lies in III quadrant.

Let $z = -1 - i\sqrt{3}$
 $= r(\cos \theta + i \sin \theta)$

Then, $x = r \cos \theta = -1$

and $y = r \sin \theta = -\sqrt{3}$

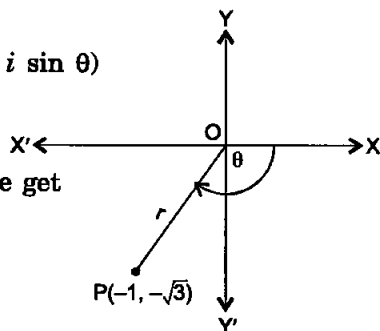
Squaring and adding both, we get

$$r^2 (\cos^2 \theta + \sin^2 \theta)$$

$$= (-1)^2 + (-\sqrt{3})^2$$

$$\Rightarrow r^2 = 1 + 3 = 4$$

$$\therefore r = 2$$



(or $r = |z| = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$)

Because $P(x, y) = (-1, -\sqrt{3})$ lies in third quadrant,

therefore, $\arg z$ is to be of the form $\theta - \pi$

$$\Rightarrow \tan \theta = \frac{y}{x} = \frac{-\sqrt{3}}{-1} = \sqrt{3} = \tan \frac{\pi}{3}$$

$$= \tan \left(\frac{\pi}{3} - \pi \right) = \tan \left(\frac{-2\pi}{3} \right)$$

$$\Rightarrow \theta = -\frac{2\pi}{3}$$

$$\therefore |z| = r = 2$$

and $\arg z = \theta = -\frac{2\pi}{3}$.

2. $z = -\sqrt{3} + i$

Sol. $z = -\sqrt{3} + i = x + iy$

is represented by the point

$P(-\sqrt{3}, 1)$ which lies in second quadrant.

Let $z = -\sqrt{3} + i$
 $= r(\cos \theta + i \sin \theta)$

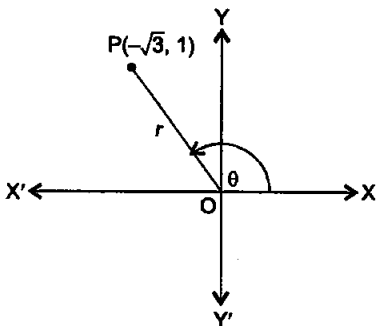
Then, $x = r \cos \theta = -\sqrt{3}$

and $y = r \sin \theta = 1$

Squaring and adding both, we get

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 3 + 1$$

$$\Rightarrow r^2 = 4 \quad \therefore r = 2$$



$$\text{(or } r = |z| = \sqrt{x^2 + y^2} = \sqrt{(-\sqrt{3})^2 + (1)^2} = \sqrt{3+1} = \sqrt{4} = 2)$$

Because $P(x, y) = (-\sqrt{3}, 1)$ lies in second quadrant, therefore $\arg z$ is to be of the form $\pi - \theta$

$$\begin{aligned} \tan \theta &= \frac{y}{x} = -\frac{1}{\sqrt{3}} = -\tan \frac{\pi}{6} \\ &= \tan \left(\pi - \frac{\pi}{6} \right) = \tan \frac{5\pi}{6} \end{aligned}$$

$$\Rightarrow \theta = \frac{5\pi}{6}$$

$$\therefore |z| = r = 2 \text{ and } \arg z = \theta = \frac{5\pi}{6}.$$

Convert each of the complex numbers given in Exercises 3 to 8 in the polar form:

3. $1 - i$

Sol. Let $z = 1 - i = x + iy$ is represented by the point $P(x, y) = (1, -1)$ which lies in fourth quadrant.

$$\text{Let } z = 1 - i = r(\cos \theta + i \sin \theta) \quad \dots (i)$$

$$\text{then } x = 1 = r \cos \theta \text{ and } y = -1 = r \sin \theta$$

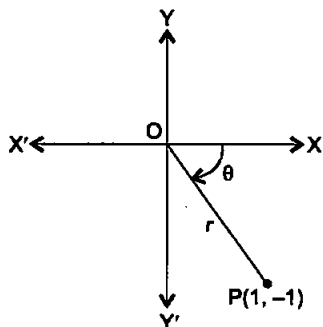
Squaring and adding both,

$$\therefore 1^2 + (-1)^2 = r^2 \Rightarrow r = \sqrt{2}$$

$$\text{(or } r = |z| = \sqrt{x^2 + y^2} =$$

$$\sqrt{(1)^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2})$$

Because $P(1, -1)$ lies in fourth quadrant therefore, $\arg z$ is to be of the form $-\theta$.



$$\tan \theta = \frac{y}{x} = \frac{-1}{1} = -1 = -\tan \frac{\pi}{4} = \tan \left(-\frac{\pi}{4} \right)$$

$$\therefore \theta = -\frac{\pi}{4}$$

Putting these values of r and θ in (i),

the required polar form of z is $\sqrt{2} \left[\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right]$.

4. $-1 + i$

Sol. $z = -1 + i = x + iy$ is represented by the point $P(x, y) = P(-1, 1)$ which lies in second quadrant.

Let $z = -1 + i = r(\cos \theta + i \sin \theta)$... (i)

Then, $r \cos \theta = -1 = x$

and $r \sin \theta = 1 = y$

Squaring and adding both, we get

$$r^2 (\cos^2 \theta + \sin^2 \theta)$$

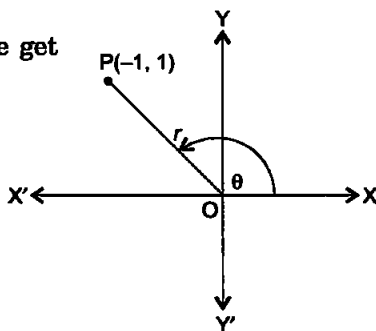
$$= (-1)^2 + 1^2$$

$$\Rightarrow r^2 = 1 + 1 = 2$$

$$\therefore r = \sqrt{2}$$

$$\text{(or } r = |z| = \sqrt{x^2 + y^2}$$

$$= \sqrt{(-1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2})$$



Because $P(-1, 1)$ lies in second quadrant, therefore $\arg z$ is to be of the form $\pi - \theta$.

$$\tan \theta = \frac{y}{x} = -1 = -\tan \frac{\pi}{4} = \tan \left(\pi - \frac{\pi}{4} \right) = \tan \frac{3\pi}{4}$$

$$\therefore \theta = \frac{3\pi}{4}$$

Putting these values of r and θ in (i),

$z = \sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$ is the required polar form.

5. $-1 - i$

Sol. $z = -1 - i = x + iy$ is represented by the point $P(x, y) = (-1, -1)$ which lies in third quadrant.

Let $z = -1 - i$

$$= r(\cos \theta + i \sin \theta) \quad \dots (i)$$

Then, $r \cos \theta = -1 (= x)$

and $r \sin \theta = -1 (= y)$

Squaring and adding both, we get

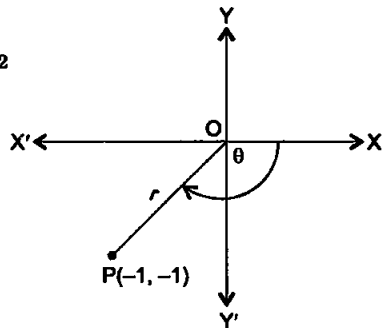
$$r^2 (\cos^2 \theta + \sin^2 \theta) = (-1)^2 + (-1)^2$$

$$\Rightarrow r^2 = 1 + 1 = 2$$

$$\therefore r = \sqrt{2}$$

$$\text{(or } r = |z| = \sqrt{x^2 + y^2}$$

$$= \sqrt{(-1)^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2})$$



Because P (-1, -1) lies in third quadrant, therefore arg z is to be of the form $-(\pi - \theta) = \theta - \pi$

$$\tan \theta = \frac{y}{x} = 1 = \tan \frac{\pi}{4} = \tan (\theta - \pi)$$

$$= \tan \left(\frac{\pi}{4} - \pi \right)$$

$$= \tan \left(-\frac{3\pi}{4} \right)$$

$$\therefore \theta = -\frac{3\pi}{4}$$

Putting these values of r and θ in (i),

$$z = \sqrt{2} \left[\cos \left(-\frac{3\pi}{4} \right) + i \sin \left(-\frac{3\pi}{4} \right) \right] \text{ is the}$$

required polar form.

6. - 3

Sol. $z = -3 = -3 + 0i = x + iy$ is represented by the point P (x, y) = (-3, 0) which lies on the ray OX'.

Let $z = -3 = r (\cos \theta + i \sin \theta)$... (i)

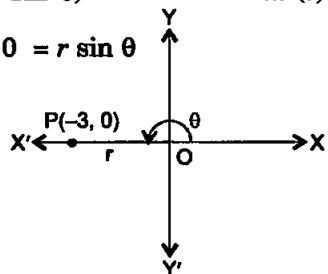
Then $x = -3 = r \cos \theta$, and $y = 0 = r \sin \theta$

Squaring and adding both,

$$\therefore (-3)^2 + 0^2 = r^2 \Rightarrow r = 3$$

$$\text{(or } r = |z| = \sqrt{x^2 + y^2} =$$

$$\sqrt{(-3)^2 + 0^2} = \sqrt{9+0} = \sqrt{9} = 3)$$



Because P (-3, 0) lies on OX' i.e. in second quadrant, therefore arg z is to be of the form $\pi - \theta$.

$$\tan \theta = \frac{y}{x} = \frac{0}{-3} = 0 = \tan 0 = \tan (\pi - 0) = \tan \pi$$

$$\therefore \theta = \pi$$

Putting these values of r and θ in (i), required polar form of $z = 3 (\cos \pi + i \sin \pi)$

Remark: Because P (-3, 0) lies on OX', we can also say that P lies in third quadrant and $\arg z = -\pi + \theta = -\pi + 0 = -\pi$ which is rejected because by result No. 12 of "Lesson at a glance" page 118,

$$-\pi < \theta \leq \pi \text{ i.e. } \arg z \neq -\pi.$$

7. $\sqrt{3} + i$

Sol. $z = \sqrt{3} + i = x + iy$ is represented by the point P(x, y) = P ($\sqrt{3}$, 1) which lies in first quadrant.

Let $z = \sqrt{3} + i = r (\cos \theta + i \sin \theta)$... (i)

Then, $r \cos \theta = \sqrt{3} = x$

and $r \sin \theta = 1 = y$

Squaring and adding both,

we get

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 3 + 1$$

$$\Rightarrow r^2 = 4 \therefore r = 2$$

Because the point P ($\sqrt{3}$, 1)

lies in first quadrant,

therefore, arg z is to be of the form θ .

$$\tan \theta = \frac{y}{x} = \frac{1}{\sqrt{3}} = \tan \frac{\pi}{6}$$

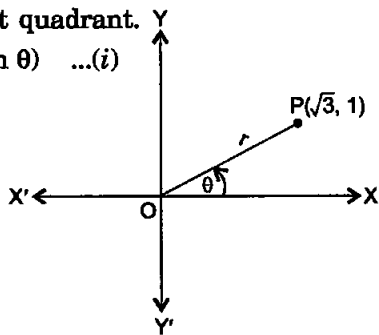
$$\therefore \theta = \frac{\pi}{6}$$

Putting values of r and θ in (i),

$$z = 2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \text{ is the required polar form.}$$

8. i

Sol. $z = i = 0 + i = x + iy$ is represented by the point P (x, y) = (0, 1) which lies on OY.



Let $z = 0 + i = r (\cos \theta + i \sin \theta)$... (i)

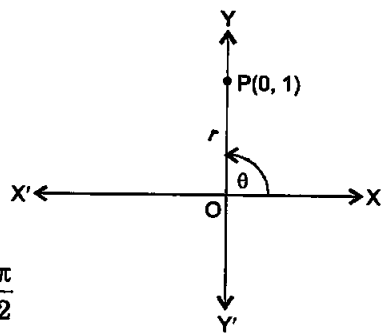
Then, $r \cos \theta = 0$ and $r \sin \theta = 1$.

Squaring and adding both, we get

$$r^2 (\cos^2 \theta + \sin^2 \theta) = 0 + 1$$

$$\Rightarrow r^2 = 1 \therefore r = 1$$

Because the point P (0, 1) lies on OY, therefore arg z is to be of the form θ .



$$\tan \theta = \frac{y}{x} = \frac{1}{0} = \infty = \tan \frac{\pi}{2}$$

$$\therefore \theta = \frac{\pi}{2}$$

Putting these values of r and θ in (i),

$$z = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \text{ is the required polar form.}$$

EXERCISE 5.3 (Page No.: 109)

Solve each of the following equations:

1. $x^2 + 3 = 0$

Sol. $x^2 + 3 = 0$

$$\Rightarrow x^2 = -3 \Rightarrow x = \pm \sqrt{-3} = \pm \sqrt{3(-1)}$$

$$\Rightarrow x = \pm \sqrt{3} \cdot \sqrt{-1}$$

$$\therefore x = \pm \sqrt{3} i.$$

2. $2x^2 + x + 1 = 0$

Sol. $2x^2 + x + 1 = 0$

Here, $a = 2, b = 1, c = 1$

$$D = b^2 - 4ac = 1 - 4 \times 2 \times 1 = 1 - 8 = -7$$

$$\therefore x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times 2} = \frac{-1 \pm \sqrt{7}i}{4}$$

3. $x^2 + 3x + 9 = 0$

Sol. $x^2 + 3x + 9 = 0$

Here, $a = 1, b = 3, c = 9$

$$D = b^2 - 4ac = 3^2 - 4 \times 1 \times 9 = 9 - 36 = -27$$

$$\therefore x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-3 \pm \sqrt{-27}}{2 \times 1} = \frac{-3 \pm 3\sqrt{3}i}{2}$$

4. $-x^2 + x - 2 = 0$

Sol. $-x^2 + x - 2 = 0$

Here, $a = -1, b = 1, c = -2$

$$D = b^2 - 4ac = 1^2 - 4 \times (-1) \times (-2) \\ = 1 - 8 = -7$$

$$\therefore x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times (-1)} = \frac{-1 \pm \sqrt{7}i}{-2}$$

5. $x^2 + 3x + 5 = 0$

Sol. $x^2 + 3x + 5 = 0$

Here, $a = 1, b = 3, c = 5$

$$D = b^2 - 4ac = 3^2 - 4 \times 1 \times 5 \\ = 9 - 20 = -11$$

$$\therefore x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-3 \pm \sqrt{-11}}{2 \times 1} = \frac{-3 \pm \sqrt{11}i}{2}$$

6. $x^2 - x + 2 = 0$

Sol. $x^2 - x + 2 = 0$

Here, $a = 1, b = -1, c = 2$

$$D = b^2 - 4ac = (-1)^2 - 4 \times 1 \times 2 = 1 - 8 = -7$$

$$\therefore x = \frac{-b \pm \sqrt{D}}{2a} = \frac{1 \pm \sqrt{-7}}{2 \times 1} = \frac{1 \pm \sqrt{7}i}{2}$$

7. $\sqrt{2}x^2 + x + \sqrt{2} = 0$

Sol. $\sqrt{2}x^2 + x + \sqrt{2} = 0$

Here $a = \sqrt{2}, b = 1, c = \sqrt{2}$

$$D = b^2 - 4ac = 1 - 4 \times \sqrt{2} \times \sqrt{2} = 1 - 8 = -7$$

$$\therefore x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-7}}{2\sqrt{2}} = \frac{-1 \pm \sqrt{7}i}{2\sqrt{2}}$$

8. $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

Sol. The given equation is $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

Here $a = \sqrt{3}$, $b = -\sqrt{2}$, $c = 3\sqrt{3}$

$$D = b^2 - 4ac = (-\sqrt{2})^2 - 4(\sqrt{3})(3\sqrt{3}) \\ = 2 - 36 = -34 < 0$$

$$\therefore x = \frac{-b \pm \sqrt{D}}{2a} = \frac{\sqrt{2} \pm \sqrt{-34}}{2\sqrt{3}} \\ = \frac{\sqrt{2} \pm \sqrt{34}i}{2\sqrt{3}} = \frac{\sqrt{2}(1 \pm \sqrt{17}i)}{2\sqrt{3}} = \frac{1 \pm \sqrt{17}i}{\sqrt{6}}$$

9. $x^2 + x + \frac{1}{\sqrt{2}} = 0$

Sol. The given equation is $x^2 + x + \frac{1}{\sqrt{2}} = 0$

Here $a = 1, b = 1, c = \frac{1}{\sqrt{2}}$

$$D = b^2 - 4ac = 1^2 - 4 \times 1 \times \frac{1}{\sqrt{2}} = 1 - 2\sqrt{2} \\ = -(2\sqrt{2} - 1) < 0$$

$$\therefore x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{-(2\sqrt{2} - 1)}}{2} \\ = \frac{-1 \pm \sqrt{2\sqrt{2} - 1}i}{2}$$

10. $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$

Sol. The given equation is $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$

Here $a = 1, b = \frac{1}{\sqrt{2}}, c = 1$

$$D = b^2 - 4ac = \left(\frac{1}{\sqrt{2}}\right)^2 - 4 \times 1 \times 1 = \frac{1}{2} - 4 = -\frac{7}{2} < 0$$

$$\begin{aligned} \therefore x &= \frac{-b \pm \sqrt{D}}{2a} = \frac{-\frac{1}{\sqrt{2}} \pm \sqrt{-\frac{7}{2}}}{2 \times 1} \\ &= \frac{-\frac{1}{\sqrt{2}} \pm \sqrt{\frac{7}{2}} i}{2 \times 1} = \frac{-1 \pm \sqrt{7} i}{2\sqrt{2}} \end{aligned}$$

MISCELLANEOUS EXERCISE ON CHAPTER 5

(Page No.: 112-113)

1. Evaluate: $\left[i^{18} + \left(\frac{1}{i}\right)^{25} \right]^3$.

Sol. $i^{18} = i^{16} i^2 = (i^4)^4 i^2 = (1)^4(-1) = -1$ [$\because i^4 = 1$
and $i^2 = -1$]

$$\left(\frac{1}{i}\right)^{25} = \frac{1}{i^{25}} = \frac{1}{i^{24} \cdot i} = \frac{1}{(i^4)^6 \cdot i}$$

$$= \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} \quad [\because (i^4)^6 = 1^6 = 1]$$

$$= \frac{i}{i^2} = \frac{i}{-1} = -i = 0 - i \quad [\because i^2 = -1]$$

$$\therefore \left[i^{18} + \left(\frac{1}{i}\right)^{25} \right]^3 = (-1 - i)^3 = [-(1 + i)]^3$$

$$[\because i^{18} = i^{16} \cdot i^2 = (i^4)^4 i^2 = (1)^4(-1) = -1 \text{ and } \frac{1}{i} = -i]$$

$$= -(1 + i)^3$$

$$= -[1^3 + i^3 + 3 \cdot 1 \cdot i(1 + i)]$$

$$[\because (a + b)^3 = a^3 + b^3 + 3ab(a + b)]$$

$$= -(1 - i + 3i + 3i^2)$$

$$= -(1 + 2i - 3)$$

$$= -(-2 + 2i) = 2 - 2i.$$

2. For any two complex numbers z_1 and z_2 , prove that $\text{Re}(z_1 z_2) = \text{Re}(z_1) \text{Re}(z_2) - \text{Im}(z_1) \text{Im}(z_2)$

Sol. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Then $\operatorname{Re}(z_1) = x_1, \operatorname{Im}(z_1) = y_1; \operatorname{Re}(z_2) = x_2, \operatorname{Im}(z_2) = y_2$

$$\begin{aligned} \text{Now } z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + ix_1 y_2 + iy_1 x_2 \\ &\quad + i^2 y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2). \end{aligned}$$

$$\begin{aligned} \therefore \operatorname{Re}(z_1 z_2) &= x_1 x_2 - y_1 y_2 \\ &= \operatorname{Re}(z_1) \operatorname{Re}(z_2) - \operatorname{Im}(z_1) \operatorname{Im}(z_2). \end{aligned}$$

3. Reduce $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to the standard form.

$$\begin{aligned} \text{Sol. } &\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right) \\ &= \left[\frac{(1+i) - 2(1-4i)}{(1-4i)(1+i)}\right]\left(\frac{3-4i}{5+i}\right) \\ &= \left(\frac{1+i-2+8i}{1+i-4i-4i^2}\right)\left(\frac{3-4i}{5+i}\right) \\ &= \left(\frac{-1+9i}{1-3i+4}\right)\left(\frac{3-4i}{5+i}\right) \\ &= \frac{(-1+9i)(3-4i)}{(5-3i)(5+i)} = \frac{-3+4i+27i-36i^2}{25+5i-15i-3i^2} \\ &= \frac{-3+31i+36}{25-10i+3} = \frac{33+31i}{28-10i} \times \frac{28+10i}{28+10i} \quad \text{(Rationalising)} \\ &= \frac{924+868i+330i+310i^2}{(28)^2 - (10i)^2} \\ &= \frac{924+1198i-310}{784-100i^2} \\ &= \frac{614+1198i}{784+100} = \frac{2(307+599i)}{884} \\ &= \frac{307+599i}{442} = \frac{307}{442} + \frac{599i}{442}. \end{aligned}$$

4. If $x - iy = \sqrt{\frac{a - ib}{c - id}}$, prove that $(x^2 + y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}$.

Sol. Given, $x - iy = \sqrt{\frac{a - ib}{c - id}}$

Squaring both sides, we get

$$(x - iy)^2 = \frac{a - ib}{c - id}$$

Taking modulus on both sides,

$$\Rightarrow |(x - iy)^2| = \left| \frac{a - ib}{c - id} \right|$$

$$\Rightarrow |x - iy|^2 = \frac{|a - ib|}{|c - id|} \left[\because |z^2| = |z|^2 \text{ and } \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \right]$$

$$\Rightarrow \left(\sqrt{x^2 + y^2} \right)^2 = \frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}} \left[\because |x + iy| = \sqrt{x^2 + y^2} \right]$$

$$\Rightarrow x^2 + y^2 = \frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}}$$

Squaring both sides, we get

$$(x^2 + y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}.$$

Second Solution

Given, $x - iy = \sqrt{\frac{a - ib}{c - id}}$

Squaring both sides, we get

$$(x - iy)^2 = \frac{a - ib}{c - id} \quad \dots (i)$$

Taking conjugates on both sides of (i),
(i.e. changing i to $-i$ in (i))

$$(x + iy)^2 = \frac{a + ib}{c + id} \quad \dots (ii)$$

Multiplying Equations (i), and (ii), we have,

$$(x - iy)^2 (x + iy)^2 = \left(\frac{a - ib}{c - id} \right) \left(\frac{a + ib}{c + id} \right)$$

$$\text{or } (x - iy) (x + iy)^2 = \frac{a^2 - i^2 b^2}{c^2 - i^2 d^2} \quad \because A^2 B^2 = (AB)^2$$

$$\text{or } (x^2 - i^2 y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}$$

$$\text{or } (x^2 + y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}$$

5. Convert the following in the polar form:

$$(i) \frac{1 + 7i}{(2 - i)^2} \qquad (ii) \frac{1 + 3i}{1 - 2i}$$

Sol. (i) $\frac{1 + 7i}{(2 - i)^2} = \frac{1 + 7i}{4 + i^2 - 4i} = \frac{1 + 7i}{3 - 4i} \quad [\because i^2 = -1]$

$$= \frac{1 + 7i}{3 - 4i} \times \frac{3 + 4i}{3 + 4i}$$
$$= \frac{3 + 4i + 21i + 28i^2}{9 - 16i^2} = \frac{3 + 25i - 28}{9 + 16}$$
$$= \frac{-25 + 25i}{25} = -1 + i$$

Now reproduce solution of Q. 4 in Exercise 5.2 page 126.

$$(ii) \frac{1 + 3i}{1 - 2i} = \frac{1 + 3i}{1 - 2i} \times \frac{1 + 2i}{1 + 2i}$$
$$= \frac{1 + 2i + 3i + 6i^2}{1 - 4i^2} = \frac{1 + 5i - 6}{1 + 4}$$
$$= \frac{-5 + 5i}{5} = -1 + i$$

Now again reproduce solution of Q. 4 in Exercise 5.2 page 126.

Solve each of the equations in Exercises 6 to 9.

6. $3x^2 - 4x + \frac{20}{3} = 0$.

Sol. The given equation is $3x^2 - 4x + \frac{20}{3} = 0$

Here, $a = 3, b = -4, c = \frac{20}{3}$
 $D = b^2 - 4ac = (-4)^2 - 4 \times 3 \times \frac{20}{3}$
 $= 16 - 80 = -64 < 0$

$\therefore x = \frac{-b \pm \sqrt{D}}{2a} = \frac{4 \pm \sqrt{-64}}{6}$
 $= \frac{4 \pm \sqrt{64}i}{2 \times 3} = \frac{4 \pm 8i}{6} = \frac{2}{3} (1 \pm 2i)$.

7. $x^2 - 2x + \frac{3}{2} = 0$

Sol. $x^2 - 2x + \frac{3}{2} = 0$

Here, $a = 1, b = -2, c = \frac{3}{2}$

$D = b^2 - 4ac = (-2)^2 - 4 \times 1 \times \frac{3}{2}$
 $= 4 - 6 = -2$

$\therefore x = \frac{-b \pm \sqrt{D}}{2a} = \frac{2 \pm \sqrt{-2}}{2 \times 1}$
 $= \frac{2 \pm \sqrt{2}i}{2} = 1 \pm \frac{\sqrt{2}}{2}i$.

8. $27x^2 - 10x + 1 = 0$

Sol. $27x^2 - 10x + 1 = 0$

Here, $a = 27, b = -10, c = 1$

$D = b^2 - 4ac = (-10)^2 - 4 \times 27 \times 1$
 $= 100 - 108 = -8$

$\therefore x = \frac{-b \pm \sqrt{D}}{2a} = \frac{10 \pm \sqrt{-8}}{2 \times 27}$
 $= \frac{10 \pm 2\sqrt{2}i}{54} = \frac{5}{27} \pm \frac{\sqrt{2}}{27}i$.

9. $21x^2 - 28x + 10 = 0$

Sol. $21x^2 - 28x + 10 = 0$

Here,

$a = 21, b = -28, c = 10$

$D = b^2 - 4ac = (-28)^2 - 4 \times 21 \times 10$

$= 784 - 840 = -56$

$\therefore x = \frac{-b \pm \sqrt{D}}{2a} = \frac{28 \pm \sqrt{-56}}{2 \times 21} = \frac{28 \pm \sqrt{4(14)(-1)}}{42}$

$= \frac{28 \pm 2\sqrt{14}i}{42} = \frac{28}{42} \pm \frac{2\sqrt{14}}{42}i$

$= \frac{2}{3} \pm \frac{\sqrt{14}}{21}i.$

10. If $z_1 = 2 - i, z_2 = 1 + i$, find $\left| \frac{z_1 + z_2 + 1}{z_1 - z_2 + 1} \right|$.

Sol. $\left| \frac{z_1 + z_2 + 1}{z_1 - z_2 + 1} \right| = \left| \frac{(2 - i) + (1 + i) + 1}{(2 - i) - (1 + i) + 1} \right|$

$= \left| \frac{4}{2 - 2i} \right| = \left| \frac{2}{1 - i} \right|$

$= \frac{|2|}{|1 - i|} = \frac{|2 + i0|}{|1 - i|} \quad \left[\because \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \right]$

$= \frac{\sqrt{4+0}}{\sqrt{1+1}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$

11. If $a + ib = \frac{(x+i)^2}{2x^2+1}$, prove that $a^2 + b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$.

Sol. Given, $a + ib = \frac{(x+i)^2}{2x^2+1}$

Taking modulus on both sides,

$\Rightarrow |a + ib| = \left| \frac{(x+i)^2}{2x^2+1} \right|$

$\Rightarrow \sqrt{a^2 + b^2} = \frac{|(x+i)^2|}{|2x^2+1|} \quad \left[\because \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \right]$

$$= \frac{|x+i|^2}{2x^2+1}$$

$[\because |z^2| = |z|^2 \text{ and } 2x^2 + 1 \text{ is a positive real number}]$

$$= \frac{(\sqrt{x^2+1})^2}{2x^2+1}$$

$$\Rightarrow \sqrt{a^2+b^2} = \frac{x^2+1}{2x^2+1}$$

Squaring both sides, we get

$$a^2 + b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$$

Second Solution

Given, $a + ib = \frac{(x+i)^2}{2x^2+1}$... (i)

Taking conjugates on both sides of (i) (i.e. changing i to $-i$ in (i))

$$a - ib = \frac{(x-i)^2}{2x^2+1}$$
 ... (ii)

Multiplying Eqns (i) and (ii), we have

$$(a + ib)(a - ib) = \frac{(x+i)^2}{2x^2+1} \frac{(x-i)^2}{2x^2+1}$$

$$\text{or } a^2 - i^2 b^2 = \frac{[(x+i)(x-i)^2]}{(2x^2+1)^2} \quad \because A^2B^2 = (AB)^2$$

$$\text{or } a^2 + b^2 = \frac{(x^2 - i^2)^2}{(2x^2+1)^2}$$

$$\text{or } a^2 + b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$$

12. Let $z_1 = 2 - i$, $z_2 = -2 + i$. Find:

(i) $\operatorname{Re} \left(\frac{z_1 z_2}{z_1} \right)$

(ii) $\operatorname{Im} \left(\frac{1}{z_1 z_1} \right)$

Sol. (i)

$$\frac{z_1 z_2}{z_1} = \frac{(2-i)(-2+i)}{(2-i)}$$

$$= \frac{-4 + 2i + 2i - i^2}{2+i} = \frac{-4 + 4i + 1}{2+i}$$

$$= \frac{-3 + 4i}{2+i} \times \frac{2-i}{2-i} \quad \text{(Rationalising)}$$

$$= \frac{-6 + 3i + 8i - 4i^2}{4 - i^2}$$

$$= \frac{-6 + 11i + 4}{4 + 1} = \frac{-2 + 11i}{5}$$

$$= -\frac{2}{5} + \frac{11}{5} i.$$

$$\therefore \operatorname{Re} \left(\frac{z_1 z_2}{z_1} \right) = \operatorname{Re} \left(-\frac{2}{5} + \frac{11}{5} i \right) = -\frac{2}{5}.$$

(ii)

$$\frac{1}{z_1 z_1} = \frac{1}{(2-i)(2+i)} = \frac{1}{4 - i^2}$$

$$= \frac{1}{4 + 1} = \frac{1}{5} = \frac{1}{5} + 0i$$

$$\therefore \operatorname{Im} \left(\frac{1}{z_1 z_1} \right) = \operatorname{Im} \left(\frac{1}{5} + 0i \right) = 0.$$

13. Find the modulus and argument of the complex number $\frac{1+2i}{1-3i}$.

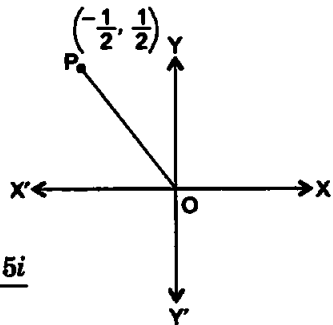
Sol.

$$\frac{1+2i}{1-3i} = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i}$$

$$= \frac{1+3i+2i+6i^2}{1-9i^2}$$

$$= \frac{1+5i-6}{1+9} = \frac{-5+5i}{10}$$

$$= -\frac{1}{2} + \frac{1}{2}i = x + iy$$



is represented by the point $P(x, y) = P\left(\frac{-1}{2}, \frac{1}{2}\right)$ which lies in second quadrant.

$$\begin{aligned} \text{Let } z &= \frac{1+2i}{1-3i} = -\frac{1}{2} + \frac{1}{2}i \\ &= r(\cos \theta + i \sin \theta). \end{aligned}$$

$$\text{Then, } r \cos \theta = -\frac{1}{2} = x \text{ and } r \sin \theta = \frac{1}{2} = y$$

Squaring and adding both, we get

$$r^2 (\cos^2 \theta + \sin^2 \theta) = \frac{1}{4} + \frac{1}{4}$$

$$\Rightarrow r^2 = \frac{1}{2} \quad \therefore r = \frac{1}{\sqrt{2}}$$

$$\begin{aligned} (\text{or } r = |z| &= \sqrt{x^2 + y^2} = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \\ &= \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}) \end{aligned}$$

Because $P\left(\frac{-1}{2}, \frac{1}{2}\right)$ lies in second quadrant, therefore $\arg z$ is to be of the form $\pi - \theta$.

$$\tan \theta = \frac{y}{x} = -1 = -\tan \frac{\pi}{4} = \tan \left(\pi - \frac{\pi}{4}\right) = \tan \frac{3\pi}{4}$$

$$\Rightarrow \theta = \frac{3\pi}{4}$$

$$\text{Hence, } |z| = r = \frac{1}{\sqrt{2}} \text{ and } \arg z = \theta = \frac{3\pi}{4}.$$

14. Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

$$\begin{aligned} \text{Sol. Given, } (x - iy)(3 + 5i) &= \overline{-6 - 24i} \\ &= -6 + 24i \end{aligned}$$

$$\begin{aligned} \Rightarrow x - iy &= \frac{-6 + 24i}{3 + 5i} \\ &= \frac{-6 + 24i}{3 + 5i} \times \frac{3 - 5i}{3 - 5i} \quad (\text{Rationalising}) \\ &= \frac{-18 + 30i + 72i - 120i^2}{9 - 25i^2} \\ &= \frac{-18 + 102i + 120}{9 + 25} \\ &= \frac{102 + 102i}{34} = \frac{34(3 + 3i)}{34} \end{aligned}$$

$$\Rightarrow x - iy = 3 + 3i$$

Equating real and imaginary parts on both sides, we have

$$x = 3 \quad \text{and} \quad -y = 3$$

$$\Rightarrow x = 3 \quad \text{and} \quad y = -3.$$

15. Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$.

Sol. Let $z = \frac{1+i}{1-i} - \frac{1-i}{1+i}$

Taking L.C.M.,

$$\begin{aligned} &= \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)} = \frac{(1+i^2+2i) - (1+i^2-2i)}{1-i^2} \\ &= \frac{1-1+2i-(1-1-2i)}{1+1} \\ &= \frac{2i+2i}{2} = \frac{4i}{2} = 2i = 0 + 2i \end{aligned}$$

$$\therefore |z| = \sqrt{0+4} = 2.$$

16. If $(x + iy)^3 = u + iv$, then show that

$$\frac{u}{x} + \frac{v}{y} = 4(x^2 - y^2).$$

Sol. Given, $u + iv = (x + iy)^3$

$$= x^3 + 3x^2 \cdot iy + 3x \cdot (iy)^2 + (iy)^3$$

$$\begin{aligned} \therefore (a + b)^3 &= a^3 + b^3 + 3ab(a + b) \\ &= a^3 + 3a^2b + 3ab^2 + b^3 \end{aligned}$$

$$= x^3 + 3ix^2y + 3i^2xy^2 + i^3y^3$$

$$= x^3 + 3ix^2y - 3xy^2 - iy^3$$

$$[\because i^2 = -1, i^3 = -i]$$

$$\Rightarrow u + iv = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

Equating real and imaginary parts on both sides, we have

$$\Rightarrow u = x^3 - 3xy^2 \quad \text{and} \quad v = 3x^2y - y^3$$

$$\Rightarrow u = x(x^2 - 3y^2) \quad \text{and} \quad v = y(3x^2 - y^2)$$

$$\Rightarrow \frac{u}{x} = x^2 - 3y^2 \quad \text{and} \quad \frac{v}{y} = 3x^2 - y^2$$

$$\text{Adding} \quad \frac{u}{x} + \frac{v}{y} = 4x^2 - 4y^2 = 4(x^2 - y^2).$$

17. If α and β are different complex numbers with $|\beta| = 1$,

$$\text{then find } \left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right|.$$

Sol. $|\beta| = 1$ (given) ... (i)

$$\therefore |\beta|^2 = 1 \quad \text{or} \quad \beta \bar{\beta} = 1$$

... (ii)

$$[\because |z|^2 = z \bar{z} \text{ for every complex number } z.]$$

$$\therefore \left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| = \left| \frac{\beta - \alpha}{\beta \bar{\beta} - \bar{\alpha}\beta} \right| \quad [\text{Putting } 1 = \beta \bar{\beta} \text{ from (ii)}]$$

$$= \left| \frac{\beta - \alpha}{\beta(\bar{\beta} - \bar{\alpha})} \right| = \left| \frac{\beta - \alpha}{\beta(\overline{\beta - \alpha})} \right|$$

$$[\because \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}]$$

$$= \frac{|\beta - \alpha|}{|\beta| |\overline{\beta - \alpha}|}$$

$$\left[\because \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \text{ and } |z_1 z_2| = |z_1| |z_2| \right]$$

$$= \frac{|\beta - \alpha|}{|\beta| |\overline{\beta - \alpha}|} \quad [\because |\bar{z}| = |z|]$$

$$= \frac{1}{|\beta|} = \frac{1}{1} \quad [\text{By (i)}]$$

$$= 1.$$

18. Find the number of non-zero integral solutions of the equation $|1 - i|^x = 2^x$.

Sol. Given, $|1 - i|^x = 2^x$... (i)

$$\Rightarrow (\sqrt{1^2 + (-1)^2})^x = 2^x$$

$$\Rightarrow (\sqrt{2})^x = 2 \quad \Rightarrow (2^{1/2})^x = 2^x$$

$$\Rightarrow 2^{x/2} = 2^x \quad \Rightarrow \frac{x}{2} = x$$

$$\Rightarrow 2x = x \quad \Rightarrow 2x - x = 0$$

$$\therefore x = 0$$

\Rightarrow The only solution of equation (i) is 0.

\Rightarrow The given equation has no non-zero integral solution.

\therefore The number of non-zero integral solutions is 0.

19. If $(a + ib)(c + id)(e + if)(g + ih) = A + iB$, then show that

$$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2.$$

Sol. Given: $(a + ib)(c + id)(e + if)(g + ih) = A + iB$

Taking modulus on both sides,

$$\Rightarrow |(a + ib)(c + id)(e + if)(g + ih)| = |A + iB|$$

$$\Rightarrow |a + ib| |c + id| |e + if| |g + ih| = |A + iB|$$

$$[\because |z_1 z_2 \dots z_n| = |z_1| |z_2| \dots |z_n|]$$

$$\Rightarrow \sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2} \cdot \sqrt{e^2 + f^2} \cdot \sqrt{g^2 + h^2} = \sqrt{A^2 + B^2}$$

Squaring both sides, we get

$$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2.$$

Second Solution.

$$\text{Given : } (a + ib)(c + id)(e + if)(g + ih) = A + iB \quad \dots (i)$$

Taking conjugates on both sides of (i),

$$(a - ib)(c - id)(e - if)(g - ih) = A - iB \quad \dots (ii)$$

Multiplying Eqns (i) and (ii) we have,

$$(a + ib)(a - ib)(c + id)(c - id)(e + if)(e - if)(g + ih)$$

$$(g - ih) = (A + iB)(A - iB)$$

$$\Rightarrow (a^2 - i^2 b^2)(c^2 - i^2 d^2)(e^2 - i^2 f^2)(g^2 - i^2 h^2)$$

$$= A^2 - i^2 B^2$$

$$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2$$

20. If $\left(\frac{1+i}{1-i}\right)^m = 1$, then find the least positive integral value of m .

Sol.
$$\left(\frac{1+i}{1-i}\right)^m = 1 \quad \Rightarrow \quad \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^m = 1$$
$$\Rightarrow \left(\frac{1+i^2+2i}{1-i^2}\right)^m = 1$$
$$\Rightarrow \left(\frac{1-1+2i}{1+1}\right)^m = 1$$
$$\Rightarrow \left(\frac{2i}{2}\right)^m = 1 \quad \Rightarrow \quad i^m = 1$$

The least positive integral value of m satisfying this equation is 4 ($\because i^4 = 1$)

