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14 Mathematical Reasoning

Lesson at a Glance

1. **Statement.** If a sentence can be judged to be either true or false (equivalently valid or invalid) but not both, is called a **statement.**

   Note. Every statement is a sentence but every sentence need not be a statement.

2. **Negation of a statement.** If p is a statement, then the negation of p denoted by $\sim p$ read as **not p** is a denial of statement p.

3. **Compound statement.** If a statement is a combination of two or more simple (smaller) statements, it is said to be a **compound statement** and the simple (smaller) statements are called **components** of the compound statement.

4(*a*). If both statements p and q are false, only then compound statement p or q is false.

(*b*) If at least one of the two component statements is true, the compound statement connected with "OR" is true.

5. If both the statements p and q are true, only then the compound statement p and q is true.

6. **De-Morgan's Laws:** If p and q are two simple statements, then

   (*a*) $\sim$ (p or q) = ($\sim$ p) and ($\sim$ q)

   (*b*) $\sim$ (p and q) = ($\sim$ p) or ($\sim$ q)

7. **Implication:** $p \Rightarrow q$ (*i.e.*, if p, then q) is false only when p is true and q is false.

8. **Contrapositive of the implication $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$.**

9. **Converse of $p \Rightarrow q$ is $q \Rightarrow p$.**

10. **Inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$.**

11(*a*). Double implication $p \Leftrightarrow q$ (p if and only if q) is true if $p \Rightarrow q$ and $q \Rightarrow p$ are both true.

(*b*) $p \Leftrightarrow q$ is true when either both p and q are true or both p and q are false.

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12. Methods used to check the validity of the statements:

(*i*) **Direct method**

Step I. Assume that p is true.

Step II. Prove that q is true.

(*ii*) **Contrapositive method**

Step I. Assume that q is false.

Step II. Then prove that p is false.

(*iii*) **Contradiction method**

Step I. Assume that p is true and q is false.

Step II. Then we arrive at some result which contradicts the above assumption. That will prove that if p is true, then q is also true.

13. Truth tables of the four connectives AND, OR, ⇒, ⇔

A table indicating the truth values of one (simple) or more (compound) statements is called a **Truth table.**

(*i*) **Truth table for "*p* and *q*"**

p	q	p and q
T	T	T
T	F	F
F	T	F
F	F	F

Note. If both p and q are true, only then 'p and q' is **true** otherwise 'p and q' is false.

(*ii*) **Truth table for "*p* or *q*"**

p	q	p or q
T	T	T
T	F	T
F	T	T
F	F	F

Note. If both p and q are false, only then 'p or q' is false otherwise 'p or q' is true.

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(*iii*) **Truth table for "$p \Rightarrow q$" (Conditional statement)**

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Note. If p is true and q is false, only then $p \Rightarrow q$ is false. In all other cases, it is true *i.e.*, **$p \Rightarrow q$ is false only when p is true q is false.**

(*iv*) **Biconditional statement**

$$p \Leftrightarrow q = (p \Rightarrow q) \text{ and } (q \Rightarrow p)$$
$$= (\text{If } p \text{ then } q) \text{ and } (\text{if } q \text{ then } p)$$
$$= p \text{ if and only if } q$$
$$= p \text{ iff } q.$$

Here p is necessary and sufficient for q.

The biconditional $p \Leftrightarrow q$ is true when both p and q are true or both are false.

EXERCISE 14.1 (*Page No.: 324*)

1. **Which of the following sentences are statements? Give reasons for your answer.**

(*i*) **There are 35 days in a month.**

Sol. This sentence is always false because the maximum number of days in a month is 31. Therefore, it is a statement.

(*ii*) **Mathematics is difficult.**

Sol. This is not a statement because for some people mathematics can be easy and for some others it can be difficult.

(*iii*) **The sum of 5 and 7 is greater than 10.**

Sol. This sentence is always true because the sum is 12 and it is greater than 10. Therefore, it is a statement.

(*iv*) **The square of a number is an even number.**

Sol. This sentence is sometimes true and sometimes not true. For example, the square of 2 is even number and the square of 3 is an odd number. Therefore, it is not a statement.

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(*v*) The sides of a quadrilateral have equal length.

Sol. This sentence is sometimes true and sometimes false. For example, squares and rhombus have sides of equal length whereas rectangles and trapezium have sides of unequal length. Therefore, it is not a statement.

(*vi*) Answer this question.

Sol. It is an order and therefore, is not a statement.

(*vii*) The product of (– 1) and 8 is 8.

Sol. This sentence is false as the product is (– 8). Therefore, it is a statement.

(*viii*) The sum of all interior angles of a triangle is 180°.

Sol. This sentence is always true and therefore, it is a statement.

(*ix*) Today is a windy day.

Sol. It is not clear from the context which day is referred and therefore, it is not a statement.

(*x*) All real numbers are complex numbers.

Sol. This sentence is always true because all real numbers x can be written in the form $x + i \times 0$ and therefore it is a statement.

2. Give three examples of sentences which are not statements. Give reasons for the answers.

Sol. The three examples can be:

(*i*) Everyone in this room is bold. This is not a statement because from the context it is not clear which room is referred here and the term bold is not precisely defined.

(*ii*) She is an engineering student. This is also not a statement because who 'she' is.

(*iii*) "$\cos^2 \theta$ is always greater than 1/2". Unless, we know what θ is, we cannot say whether the sentence is true or not.

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EXERCISE 14.2 *(Page No.: 329)*

1. Write the negation of the following statements:

(i) Chennai is the capital of Tamil Nadu.

Sol. Chennai is not the capital of Tamil Nadu.

(ii) $\sqrt{2}$ is not a complex number.

Sol. $\sqrt{2}$ is a complex number.

(iii) All triangles are not equilateral triangle.

Sol. All triangles are equilateral triangles.

(iv) The number 2 is greater than 7.

Sol. The number 2 is not greater than 7.

(v) Every natural number is an integer.

Sol. Every natural number is not an integer.

2. Are the following pairs of statements negations of each other:

(i) The number x is not a rational number.

The number x is not an irrational number.

Sol. The negation of the first statement is "the number x is a rational number", which is the same as the second statement". This is because when a number is not irrational, it is a rational. Therefore, the given pair are negations of each other.

(ii) The number x is a rational number.

The number x is an irrational number.

Sol. The negation of the first statement is "x is an irrational number" which is the same as the second statement. Therefore, the pair are negations of each other.

3. Find the component statements of the following compound statements and check whether they are true or false.

(i) Number 3 is prime or it is odd.

Sol. Component: Number 3 is prime is true

Component: Number 3 is odd is also true.

$\therefore$ The compound statement: Number 3 is prime or 3 is odd is also true.

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(ii) All integers are positive or negative.

Sol. Component p: All integers are positive is false

Component q: All integers are negative is also false.

$\therefore$ The compound statement p (OR) (AND) is false.

(iii) 100 is divisible by 3, 11 and 5.

Sol. 100 is divisible by 3 is false, 100 is divisible by 11 is false and 100 is divisible by 5 (True).

$\therefore$ The compound statement p and q and r is false.

EXERCISE 14.3 (*Page No.: 334–335*)

1. For each of the following compound statements first identify the connecting words and then break it into component statements.

(i) All rational numbers are real and all real numbers are not complex.

Sol. "And". The component statements are:

All rational numbers are real.

All real numbers are not complex.

(ii) Square of an integer is positive or negative.

Sol. "Or". The component statements are:

Square of an integer is positive.

Square of an integer is negative.

(iii) The sand heats up quickly in the Sun and does not cool down fast at night.

Sol. "And", the component statements are:

The sand heats up quickly in the sun.

The sand does not cool down fast at night.

(iv) $x = 2$ and $x = 3$ are the roots of the equation $3x^2 - x - 10 = 0$.

Sol. "And". The component statements are:

$x = 2$ is a root of the equation $3x^2 - x - 10 = 0$

$x = 3$ is a root of the equation $3x^2 - x - 10 = 0$.

2. Identify the quantifier in the following statements and write the negation of the statements.

(i) There exists a number which is equal to its square.

Sol. "There exists". The negation is

There does not exist a number which is equal to its square.

(ii) For every real number x, x is less than $x + 1$.

Sol. "For every". The negation is

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There exists a real number x such that x is not less than $x + 1$.

(*iii*) There exists a capital for every state in India.

Sol. "There exists". The negation is

There exists a state in India which does not have a capital.

Remark: 1. There are only two quantifers:

(*i*) There exists (*ii*) For all (FOR EVERY)

2. Negation of **there exists** is there does not exist.

3. For negation of 'for all', see (*ii*) above.

3. Check whether the following pair of statements are negation of each other. Give reasons for your answer.

(*i*) $x + y = y + x$ is true for every real numbers x and y.

(*ii*) There exists real numbers x and y for which $x + y = y + x$.

Sol. No. The negation of the statement in (*i*) is "There exist real number x and y for which $x + y \neq y + x$", instead of the statement given in (*ii*).

4. State whether the "Or" used in the following statements is "exclusive" or "inclusive". Give reasons for your answer.

(*i*) Sun rises or Moon sets.

Sol. Here "Or" is exclusive because only one of the two can occur. The simultaneous occurrence of the two is not possible.

(*ii*) To apply for a driving license, you should have a ration card or a passport.

Sol. Here "Or" is inclusive because you can apply for a driving licence also when you have both the ration card and the passport.

(*iii*) All integers are positive or negative.

Sol. Here "Or" is exclusive because an integer is either positive or negative. No integer can be both positive and negative.

EXERCISE 14.4 (*Page No.: 338–339*)

1. Rewrite the following statement with "if-then" in five different ways conveying the same meaning.

If a natural number is odd, then its square is also odd.

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Sol. (*i*) A natural number is odd implies that its square is odd.

(*ii*) A natural number is odd only if its square is odd.

(*iii*) For a natural number to be odd it is necessary that its square is odd.

(*iv*) For the square of a natural number to be odd, it is sufficient that the number is odd.

(*v*) If the square of a natural number is not odd, then the natural number is not odd.

Remark: If p then q means

(*i*) $p \Rightarrow q$

(*ii*) p is sufficient for q.

(*iii*) q is necessary for p.

(*iv*) $\sim q \Rightarrow \sim p$.

(*v*) p only if q.

2. Write the contrapositive and converse of the following statements.

(*i*) If x is a prime number, then x is odd.

Sol. The contrapositive for $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$.

If a number x is not odd, then x is not a prime number.

The converse for $p \Rightarrow q$ is $q \Rightarrow p$.

If a number x is odd, then it is a prime number.

(*ii*) If the two lines are parallel, then they do not intersect in the same plane.

Sol. The contrapositive for $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$.

If two lines intersect in the same plane, then they are not parallel.

The converse for $p \Rightarrow q$ is $q \Rightarrow p$.

If two lines do not intersect in the same plane, then they are parallel.

(*iii*) Something is cold implies that it has low temperature.

Sol. The contrapositive for $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$.

If something is not at low temperature, then it is not cold.

The converse for $p \Rightarrow q$ is $q \Rightarrow p$.

If something is at low temperature, then it is cold.

(*iv*) You cannot comprehend geometry if you do not know how to reason deductively.

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Sol. The contrapositive is

If you know how to reason deductively, then you can comprehend geometry.

The converse is

If you do not know how to reason deductively, then you cannot comprehend geometry.

(v) x is an even number implies that x is divisible by 4.

Sol. This statement can be written as "If x is an even number, then x is divisible by 4".

The contrapositive is, If x is not divisible by 4, then x is not an even number.

The converse is, If x is divisible by 4, then x is an even number.

3. Write each of the following statements in the form 'if-then"

(i) You get a job implies that your credentials are good.

Sol. If you get a job, then your credentials are good.

(ii) The Banana tree will bloom if it stays warm for a month.

Sol. If the banana tree stays warm for a month, then it will bloom.

(iii) A quadrilateral is a parallelogram if its diagonals bisect each other.

Sol. If diagonals of a quadrilateral bisect each other, then it is a parallelogram.

(iv) To get an A$^+$ in the class, it is necessary that you do all the exercises of the book.

Sol. If you get A$^+$ in the class, then you do all the exercises in the book.

4. Given statements in (a) and (b). Identify the statements given below as contrapositive or converse of each other.

(a) If you live in Delhi, then you have winter clothes.

(i) If you do not have winter clothes, then you do not live in Delhi.

(ii) If you have winter clothes, then you live in Delhi.

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Sol. (*i*) Contrapositive.

(*ii*) Converse.

(*b*) If a quadrilateral is a parallelogram, then its diagonals bisect each other.

(*i*) If the diagonals of a quadrilateral do not bisect each other, then the quadrilateral is not a parallelogram.

(*ii*) If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

Sol. (*i*) Contrapositive.

(*ii*) Converse.

EXERCISE 14.5 (*Page No.: 342–343*)

1. Show that the statement

p: "If x is a real number such that $x^3 + 4x = 0$, then x is 0" is true by

(*i*) direct method,

(*ii*) method of contradiction,

(*iii*) method of contrapositive.

Sol. (*i*) **Direct method:**

Let $x^3 + 4x = 0$, $x \in R$

$\Rightarrow \quad x(x^2 + 4) = 0$

$\Rightarrow$ Either $\quad x = 0$ or $x^2 + 4 = 0$

But $x^2 + 4 \geq 4$ because $x \in R$ and hence $\neq 0$.

Therefore $x = 0$

$\therefore \quad x^3 + 4x = 0, x \in R \quad \Rightarrow \quad x \neq 0.$

Thus, p is a true statement.

(*ii*) **Method of contradiction:**

Let $x^3 + 4x = 0$, $x \in R$

Suppose $x \neq 0$

$\Rightarrow \quad x^2 > 0 \quad \Rightarrow \quad x^2 + 4 > 4$

$\Rightarrow \quad x^2 + 4 \neq 0$

Now $x \neq 0$ and $x^2 + 4 \neq 0 \quad \Rightarrow \quad x(x^2 + 4) \neq 0$

$\Rightarrow \quad x^3 + 4x \neq 0$ which is a contradiction to given.

$\therefore$ Our supposition is wrong and hence $x = 0$.

Thus, p is a true statement.

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(*iii*) **Method of contrapositive:** The components of the given if...then statement p are:

Let $q : x \in$ R and $x^3 + 4x = 0$ and $r : x = 0$.

$\therefore$ The given statement p is $q \Rightarrow r$.

Its contrapositive is $\sim r \Rightarrow \sim q$.

Let $\sim r$ be true *i.e.*, x is a non-zero real number.

Now $x \neq 0, x \in$ R $\Rightarrow x^2 > 0$

$\Rightarrow x^2 + 4 > 4 \Rightarrow x^2 + 4 \neq 0$

$\therefore x(x^2 + 4) \neq 0 \Rightarrow x^3 + 4x \neq 0$

$\therefore \sim q$ is true.

i.e., $\sim r \Rightarrow \sim q$.

$\therefore q \Rightarrow r$ is true.

Thus, p is a true statement.

2. Show that the statement "For any real numbers a and b, $a^2 = b^2$ implies that $a = b$" is not true by giving a counter example.

Sol. Let $a = 2$, $b = -2$ be two real numbers.

Clearly, $a^2 = b^2$ (=4) but $a \neq b$.

Thus, the given statement is not true.

3. Show that the following statement is true by the method of contrapositive.

***p*: If x is an integer and x^2 is even, then x is also even.**

Sol. The component statements of the given statement p are:

$q : x$ is an integer such that x^2 is even.

$r : x$ is even

We have to prove, using the method of contrapositive whether $q \Rightarrow r$ is true.

[*i.e.*, its contrapositive $\sim r \Rightarrow \sim q$ is true]

Let $\sim r$ be true *i.e.*, r be false.

i.e., Let us assume that x is not even (integer) *i.e.*, x is odd.

$\therefore x = 2m + 1$ where m is an integer.

$\therefore x^2 = (2m + 1)^2 = 4m^2 + 1 + 4m$

$= 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$

$= 2t + 1$ where $t = 2m^2 + 2m$ is an integer.

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$\Rightarrow x^2$ is also odd $\Rightarrow$ x is not even.

$\Rightarrow$ q is false. [By def. of q] $\Rightarrow$ $\sim q$ is true.

$\therefore$ $\sim r \Rightarrow \sim q$ is true.

Thus, by the method of contrapositive $q \Rightarrow r$ is true.

4. By giving a counter example, show that the following statements are not true.

(i) p: If all the angles of a triangle are equal, then the triangle is an obtuse angled triangle.

Sol. Consider a triangle (equilateral) ABC in which A = B = C = 60°, then the triangle ABC is not obtuse angled though all its angles are equal.

$\therefore$ The statement p is not true.

(ii) q: The equation $x^2 - 1 = 0$ does not have a root lying between 0 and 2.

Sol. 1 lies between 0 and 2. Also, 1 satisfies the equation $x^2 - 1 = 0$. $(\because \; 1^2 - 1 = 1 - 1 = 0)$

$\therefore$ The statement q is not true.

5. Which of the following statements are true and which are false? In each case give a valid reason for saying so.

(i) p: Each radius of a circle is a chord of the circle.

Sol. False. By definition of the chord, it should intersect the circle in two points. But the radius intersects the circle only at one point.

(ii) q: The centre of a circle bisects each chord of the circle.

Sol. False. This can be shown by drawing a chord of the circle which is not a diameter.

(iii) r: Circle is a particular case of an ellipse.

Sol. True. In the equation of an ellipse if we put $b = a$, then it is a circle (Direct Method).

Putting $b = a$ in the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of the ellipse, it becomes $\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$ or $\frac{x^2+y^2}{a^2}=1$ or $x^2 + y^2 = a^2$ which is the equation of a circle.

(iv) s: If x and y are integers such that $x > y$, then $-x < -y$.

Sol. True. Given $x > y$. Multiplying by -1, $-x < -y$.

(v) t: $\sqrt{11}$ is a rational number.

Sol. False. Since 11 is a prime number, therefore $\sqrt{11}$ is irrational.

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MISCELLANEOUS EXERCISE ON CHAPTER 14

(Page No.: 345)

1. Write the negation of the following statements:

(i) p: **For every positive real number x, the number $x - 1$ is also positive.**

Sol. There exists a positive real number x such that $x - 1$ is not positive.

(ii) q: **All cats scratch.**

Sol. There exists a cat which does not scratch.

(iii) r: **For every real number x, either $x > 1$ or $x < 1$.**

Sol. There exists a real number x such that neither $x > 1$ nor $x < 1$.
($\because$ By De-Morgan's Law; $\sim (p \text{ or } q) = \sim p \text{ and } \sim q$).

(iv) s: **There eixsts a number x such that $0 < x < 1$.**

Sol. There does not exist a number x such that $0 < x < 1$.

2. State the converse and contrapositive of each of the following statements:

(i) p: **A positive integer is prime only if it has no divisors other than 1 and itself.**

Sol. The statement can be written as "If a positive integer is prime, then it has no divisors other than 1 and itself."
[$\because$ only if means "if ... then".]
See Remark(v), Q.1., Exercise 14.4, Page 446.
The converse of the statement is
If a positive integer has no divisors other than 1 and itself, then it is a prime.
The contrapositive of the statement is
If positive integer has divisors other than 1 and itself then it is not prime.

(ii) q: **I go to a beach whenever it is a sunny day.**

Sol. The given statement can be written as "If it is a sunny day, then I go to a beach".
The converse of the statement is
If I go to beach, then it is a sunny day.
The contrapositive is
If I do not go to a beach, then it is not a sunny day.

(iii) r: If it is hot outside, then you feel thirsty.

Sol. The converse is
If you feel thirsty, then it is hot outside.

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The contrapositive is

If you do not feel thirsty, then it is not hot outside.

3. Write each of the statements in the form "if *p*, then *q*"

(*i*) *p*: It is necessary to have a password to log on to the server.

Sol. If you log on the server, then you have a password.

(*ii*) *q*: There is traffic jam whenever it rains.

Sol. If it rains, then there is traffic jam.

(*iii*) *r*: You can access the website only if you pay a subscription fee.

Sol. If you can access the website, then you pay a subscription fee.

4. Rewrite each of the following statements in the form "*p* if and only if *q*"

(*i*) *p*: If you watch television, then your mind is free and if your mind is free, then you watch television.

Sol. You watch television if and only if your mind in free.

(*ii*) *q*: For you to get an A grade, it is necessary and sufficient that you do all the homework regularly.

Sol. You get an A grade if and only if you do all the homework regularly.

(*iii*) *r*: If a quadrilateral is equiangular, then it is a rectangle and if a quadrilateral is a rectangle, then it is equiangular.

Sol. A quadrilateral is equiangular if and only if it is a rectangle.

5. Given below are two statements.

p: 25 is a multiple of 5.

q: 25 is a multiple of 8.

Write the compound statements connecting these two statements with "And" and "Or". In both cases, check the validity of the compound statement.

Sol. The compound statement with "And" is 25 is a multiple of 5 and 8.

This is a false statement since

Component q: 25 is a multiple of 8 is false and connective is "And".

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Note that a compound statement with connective "And" is true only when each component statement is true, otherwise it is false.

The compound statement with "Or" is 25 is a multiple of 5 or 8.

This is a true statement since p is true.

Note that a compound statement with connective "Or" is true even when any one component statement is true and false when every component statement is false.

6. Check the validity of the statements given below by the method given against it.

(i) p: The sum of an irrational number and a rational number is irrational (by contradiction method).

(ii) q: If n is a real number with $n > 3$, then $n^2 > 9$ (by contradiction method).

Sol. (i) Let x be an irrational number and y be a rational number. If possible, let $x + y$ be a rational number.

Let $x + y = r$; then r is rational.

$\Rightarrow \quad x = r - y$

$\Rightarrow \quad x$ is rational

($\because$ difference of two rational numbers is rational)

which is a contradiction, since x is irrational (given).

$\Rightarrow$ Our supposition is wrong.

Hence $x + y$ is irrational.

(ii) Given: n is a real number with $n > 3$

If possible: let n^2 be not greater than 9.

$\because n^2 \le 9$

Now, $\quad n^2 \le 9 \quad \Rightarrow \quad n^2 - 9 \le 0$

$\Rightarrow \quad (n + 3)(n - 3) \le 0$

Dividing by $n - 3$ which is positive

($\because \; n > 3 \;\Rightarrow\; n - 3 > 0$)

$n + 3 \le 0$

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$\Rightarrow \quad n \le -3$

which is a contradiction, since $n > 3$.

$\Rightarrow$ Our supposition is wrong.

Hence, if n is real with $n > 3$, then $n^2 > 9$.

7. Write the following statement in five different ways, conveying the same meaning.

***p*: If a triangle is equiangular, then it is an obtuse angled triangle.**

Sol. The five different ways are:

(*i*) A triangle is equiangular implies that it is obtuse angled.

(*ii*) A triangle is equiangular only if it is obtuse angled.

(*iii*) For a triangle to be equiangular, it is necessary that it is obtuse angled.

(*iv*) For a triangle to be obtuse angled, it is sufficient that it is equiangular.

(*v*) If a triangle is not obtuse angled, then it is not equiangular.

Note: See "Remark", Q.1., Exercise 14.4, Page 446.

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